# Generating dimension formulas for multivariate splines


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**Abstract**

Dimensions of spaces of multivariate splines remain unknown in general. A computational method to obtain explicit formulas for the dimension of spline spaces on simplicial partitions is described. The method is based on Hilbert series and Hilbert polynomials. It is applied to conjecture the dimension formulas for splines on the Alfeld split of a simplex and on several other partitions.



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## 1 Introduction

Let $\Delta_n$ denote a simplicial partition of a polyhedral domain $\Omega \subseteq \mathbb{R}^n$, so that if any two simplices in $\Delta_n$ intersect, then their intersection is a facet of $\Delta_n$. The space of $\mathcal{C}^r$ splines of degree $\leq d$ in $n$ variables on $\Delta_n$ is

$$\mathcal{S}_d^r(\Delta_n) := \big\{ s \in \mathcal{C}^r(\Omega) : \ s|_T \in \mathcal{P}_{d,n} \text{ for each simplex } T \in \Delta_n \big\},$$

where $\mathcal{P}_{d,n}$ is the space of polynomials of degree $\leq d$ in $n$ variables. We are interested in the dimension of the space $\mathcal{S}_d^r(\Delta_n)$. For fixed $d$ and $r$, determining a closed formula for arbitrary partitions is still a major open problem, even in the bivariate case, see [9]. In this case, it is known [9, p. 240] that if $\Delta_2$ is a shellable (regular with no holes) triangulation, then

(1) $$\dim \mathcal{S}_d^r(\Delta_2) \geq \binom{d+2}{2} + E_I \binom{d+1-r}{2} - V_I \left[ \binom{d+2}{2} - \binom{r+2}{2} \right] + \sum_{v \in \mathcal{V}_I} \sigma_v,$$


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where $E_I$ is the number of interior edges, $V_I$ is the number of interior vertices, $\mathcal{V}_I$ is the set of interior vertices of $\Delta_2$, and

$$\sigma_v := \sum_{j=1}^{d-r} \max\{r+j+1-jm_v, 0\}, \quad m_v := \text{number of different edge slopes meeting at } v.$$

The right-hand side of (1) is the correct expression for the dimension if $d \geq 3r+1$, see [9, p.247 and p.273]. Not much is known for $d \leq 3r$, and it is somewhat staggering that the dimensions of $\mathcal{S}_3^1(\Delta_2)$ and of $\mathcal{S}_2^1(\Delta_2)$ remain uncertain in general. Let us point out that the right-hand side of inequality (1) can be rewritten as a linear combination of binomial coefficients – a form that is favored in this paper:

$$\begin{aligned}\dim \mathcal{S}_d^r(\Delta_2) \geq & \binom{d+2}{2} + (E_I - 3V_I)\binom{d+1-r}{2} \\ & + V_I\binom{d+1-\mu}{2} + V_I\binom{d+1-\nu}{2} + \Big(3V_I - \sum_{v\in\mathcal{V}_I} m_v\Big)\binom{\mu+1-r}{2},\end{aligned}$$

where

$$\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor, \quad \nu := r + \left\lceil \frac{r+1}{2} \right\rceil.$$

In the case of a cell $C_2$ – a triangulation with one interior vertex $v$ – it is known that the lower bound is the correct dimension, namely

$$\dim \mathcal{S}_d^r(C_2) = \binom{d+2}{2} + (E_I-3)\binom{d+1-r}{2} + \binom{d+1-\mu}{2} + \binom{d+1-\nu}{2} + (3-m)\binom{\mu+1-r}{2},$$

where $m \leq E_I$ is the number of different slopes of the $E_I$ interior edges meeting at $v$. In fact, the formula for the cell is the basis of the argument used to derive (1). This example demonstrates that the dimension depends not only on the combinatorics of $\Delta_n$ – number of vertices, edges, and other faces – but also on its exact geometry. The point of view adopted in this paper consists in fixing the partition and looking for dimension formulas valid for all $d$, $r$, and possibly $n$. The main experimental result, namely Conjecture 1, concerns the spline spaces on the Alfeld split of a single simplex. This split is a generalization of the Clough–Tocher split of a triangle to higher spacial dimensions. The Clough–Tocher split of a triangle has one interior vertex, three interior edges, and three subtriangles. The split of a tetrahedron with one interior vertex, four interior edges, six interior faces, and four subtetrahedra was introduced in [2]. We shall refer to the split of a simplex in $\mathbb{R}^n$ with $\binom{n+1}{k}$ interior $k$-dimensional faces, $0 \leq k \leq n$, as the Alfeld split $A_n$. The following is our conjecture on the dimension.

**Conjecture 1.** The dimension of the space $\mathcal{S}_d^r(A_n)$ of splines of degree $\leq d$ in $n$ variables over the Alfeld split $A_n$ of a simplex is given by

$$\dim \mathcal{S}_d^r(A_n) = \binom{d+n}{n} + \begin{cases} n\binom{d+n-\frac{r+1}{2}(n+1)}{n}, & \text{if } r \text{ is odd}, \\ \binom{d+n-1-\frac{r}{2}(n+1)}{n} + \cdots + \binom{d-\frac{r}{2}(n+1)}{n}, & \text{if } r \text{ is even}. \end{cases}$$

This formula was obtained using the computational method that we introduce in Section 2. In Section 3, we describe the steps leading to Conjecture 1, and report without details other formulas obtained via this method for several tetrahedral partitions. In Section 4, we discuss the potential of the method.

## 2 The computational method

In this section, we show how to derive an explicit formula for the dimension of $\mathcal{S}_d^r(\Delta_n)$, in the form of a linear combination of binomial coefficients, using computed values of this dimension for a finite number of parameters $r$ and $d$. We first show why the sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ depends only on a finite number of its values.

Let us for now fix the number $n$ of variables, the simplicial partition $\Delta_n$, and the smoothness parameter $r$. It is well-known that the dimension of $\mathcal{S}_d^r(\Delta_n)$ agrees with a polynomial of degree $n$ in variable $d$ when $d$ is sufficiently large. This polynomial is called the Hilbert polynomial, and it is denoted by $H := H_{\Delta_n,r}$ throughout this paper. We denote by $d^\star := d^\star_{\Delta_n,r}$ the smallest integer such that

$$\dim \mathcal{S}_d^r(\Delta_n) = H(d) \quad \text{for all } d \geq d^\star.$$

The sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ is determined by its first $d^\star + n + 1$ values. Indeed, the terms

$$\{\dim \mathcal{S}_d^r(\Delta_n),\ d^\star \leq d \leq d^\star + n\}$$

define $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq d^\star}$ by interpolation of the Hilbert polynomial, while the values

$$\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \leq d \leq d^\star - 1\}$$

complete the first $d^\star$ terms of the sequence. The estimation of $d^\star$ remains a key question. Our method incorporates the widely accepted assumption that

$$d^\star_{\Delta_n,r} \leq r2^n + 1. \tag{2}$$

This is suggested by the technique of partitioning the minimal determining set into non-overlapping subsets associated with each face, see [4]. Moreover, for the subspace of $\mathcal{S}_d^r(\Delta_n)$ imposing additional (or super) smoothness $r2^{n-j-1}$ across every $j$-dimensional face of $\Delta_n$, it was shown in [5] that the dimension is indeed a polynomial in $d$ for $d \geq r2^n + 1$. The bound (2) is likely to be an overestimation, though. The examples of Section 3 and the improved bound $d^\star_{\Delta_2,r} \leq 3r + 2$ obtained in [8] for shellable triangulations supports this belief. Reducing the bound would reduce the number of dimension values to be computed. Since splines with degrees not exceeding smoothness are simply polynomials, we have

$$\dim \mathcal{S}_d^r(\Delta_n) = \binom{d+n}{n} \quad \text{for } d \leq r.$$

Thus, assuming (2), only the $r(2^n-1)+n+1$ values $\{\dim \mathcal{S}_d^r(\Delta_n),\ r+1\le d\le r2^n+n+1\}$ are left to be computed. An additional saving can be made by using the values for smaller degrees, since we have

$$\left[\dim \mathcal{S}_d^r(\Delta_n)=\dim \mathcal{P}_{d,n}=\binom{d+n}{n}\right]\Longrightarrow\left[\dim \mathcal{S}_k^r(\Delta_n)=\dim \mathcal{P}_{k,n}=\binom{k+n}{n}\ \text{for } k\le d\right].$$

Assuming that computing $\dim \mathcal{S}_d^r(\Delta_n)$ is possible for any $d\ge 0$, the above described method gives us access to the whole sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\ge 0}$. To obtain an explicit formula, we rely on the concept of Hilbert series, i.e., the generating function of the sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\ge 0}$. According to [6, Theorem 2.8], it satisfies

$$\sum_{d\ge 0}\dim \mathcal{S}_d^r(\Delta_n)\, z^d=\frac{P(z)}{(1-z)^{n+1}}, \tag{3}$$

for some polynomial $P:=P_{\Delta_n,r}$ with integer coefficients. Denoting these coefficients by $a_k=a_{k,\Delta_n,r}$, and denoting the degree of $P$ by $k^\star=k^\star_{\Delta_n,r}$, that is,

$$P(z)=\sum_{k=0}^{k^\star}a_k z^k,\qquad a_{k^\star}\neq 0,$$

two further particulars are established in [6, Theorem 4.5]:

$$P(1)=\sum_{k=0}^{k^\star}a_k=N,\qquad P'(1)=\sum_{k=0}^{k^\star}ka_k=(r+1)\,F^{\mathrm{int}}, \tag{4}$$

where $N$ and $F^{\mathrm{int}}$ represent the number of simplices and interior facets of $\Delta_n$, respectively. In the particular case when $\Delta_n$ is a single simplex, the space $\mathcal{S}_d^r(\Delta_n)$ is just the space $\mathcal{P}_{d,n}$ of polynomials of degree $d$ in $n$ variables. Then it can be seen that $P=1$ from the identity

$$\sum_{d\ge 0}\binom{d+n}{n}z^d=\frac{1}{(1-z)^{n+1}}. \tag{5}$$

This identity is clear for $n=0$ and is inductively obtained by successive differentiations with respect to $z$ for $n\ge 1$. While the derivation of the polynomial $P$ from the dimensions $\dim \mathcal{S}_d^r(\Delta_n)$ was straightforward, identity (5) conversely provides an explicit formula for the dimensions $\dim \mathcal{S}_d^r(\Delta_n)$ in terms of the coefficients of $P$. Indeed, the formula

$$\dim \mathcal{S}_d^r(\Delta_n)=\sum_{k=0}^{k^\star}a_k\binom{d+n-k}{n} \tag{6}$$

was isolated in [6] and it also follows from

$$\sum_{d\ge 0}\dim \mathcal{S}_d^r(\Delta_n)\, z^d=\sum_{k=0}^{k^\star}a_k\frac{z^k}{(1-z)^{n+1}}=\sum_{k=0}^{k^\star}\sum_{d\ge 0}a_k\binom{d+n}{n}z^{d+k}=\sum_{d\ge 0}\sum_{k=0}^{k^\star}a_k\binom{d+n-k}{n}z^d$$

by identifying the coefficients in front of each $z^d$. Taking into account that

$$\binom{d+n-k}{n} = \begin{cases} \dfrac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}, & \text{if } d \geq k, \\ 0 = \dfrac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}, & \text{if } k-n \leq d \leq k-1, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad , & \text{if } d \leq k-n-1, \end{cases}$$

we observe that, for $d \geq k^\star - n$, the dimension of $\mathcal{S}_d^r(\Delta_n)$ agrees with the Hilbert polynomial

$$H(d) := \sum_{k=0}^{k^\star} a_k \frac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}.$$

Moreover, for $d = k^\star - n - 1$, we have

$$H(k^\star - n - 1) - \dim \mathcal{S}_{k^\star-n-1}^r(\Delta_n) = a_{k^\star} \left( \frac{(-1)(-2)\cdots(-n)}{n!} - 0 \right) = (-1)^n a_{k^\star} \neq 0.$$

The definition of $d^\star$ therefore yields $d^\star = k^\star - n$, and consequently, we see that

$$k^\star = d^\star + n.$$

This was intuitively anticipated because the determination of the sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ requires $d^\star + n + 1$ pieces of information while the equivalent determination of the polynomial $P$ requires the $k^\star + 1$ pieces of information corresponding to its coefficients. Now we describe a practical way to determine these coefficients from the computed values $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d=0}^{d^\star+n}$. It is simply based on the observation that

$$\begin{aligned} a_k &= \frac{1}{k!}\frac{d^k P(z)}{dz^k}|_{z=0} = \frac{1}{k!}\frac{d^k}{dz^k}\Big((1-z)^{n+1}\sum_{d\geq 0}\dim \mathcal{S}_d^r(\Delta_n)\, z^d\Big)|_{z=0} \\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{d^{k-\ell}}{dz^{k-\ell}}\Big((1-z)^{n+1}\Big)|_{z=0}\,\frac{d^\ell}{dz^\ell}\Big(\sum_{d\geq 0}\dim \mathcal{S}_d^r(\Delta_n)\, z^d\Big)|_{z=0} \\ &= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}\frac{(n+1)!}{(n+1-k+\ell)!}\,\ell!\dim \mathcal{S}_\ell^r(\Delta_n) \\ &= \sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+1}{k-\ell}\dim \mathcal{S}_\ell^r(\Delta_n). \end{aligned} \tag{7}$$

In particular, the value $\dim \mathcal{S}_0^r(\Delta_n) = 1$ yields $a_0 = 1$, then the value of $\dim \mathcal{S}_1^r(\Delta_n)$ yields $a_1$, the values of $\dim \mathcal{S}_1^r(\Delta_n)$ and of $\dim \mathcal{S}_2^r(\Delta_n)$ yield $a_2$ and so on. This shows that the computation of the coefficients $a_k$ can be performed sequentially, along with the computation of the dimensions $\dim \mathcal{S}_k^r(\Delta_n)$. As long as $\dim \mathcal{S}_k^r(\Delta_n)$ equals $\binom{k+n}{n}$, identity (5) ensures that the coefficients $a_k$ agree with the coefficients of the constant polynomial $P = 1$:

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = 0, \quad \cdots, \quad a_{d_\star} = 0,$$

where $d_\star$ denotes the largest integer such that $\dim \mathcal{S}^r_{d_\star}(\Delta_n) = \binom{d_\star+n}{n}$. As a matter of fact, applying (7) to a partition consisting of a single simplex, we obtain

$$0 = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{n+1}{k-\ell} \binom{\ell+n}{n}, \quad k \geq 1.$$

We may therefore also express the coefficient $a_k$ as

$$a_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{n+1}{k-\ell} \delta^r_\ell(\Delta_n), \quad k \geq 1, \tag{8}$$

where $\delta^r_\ell(\Delta_n)$ is the codimension of the polynomial space $\mathcal{P}_{\ell,n}$ in the spline space $\mathcal{S}^r_\ell(\Delta_n)$, i.e.,

$$\delta^r_\ell(\Delta_n) := \dim \mathcal{S}^r_\ell(\Delta_n) - \binom{\ell+n}{n},$$

which is less costly to compute than the dimension of $\mathcal{S}^r_\ell(\Delta_n)$. We finally note that at most $\min\{n+2, k-d_\star\}$ nonzero terms enter the sum in (8), since the summand is nonzero only when $\ell \geq k-n-1$ and $\ell \geq d_\star + 1$.

The computational method described above exploits the specific form of the Hilbert series. As a conclusion to this section, we make the side observation that (3) can be derived by simple means. It suffices to set $u_d = \dim \mathcal{S}^r_d(\Delta_n)$ in the following lemma.

**Lemma 1.** Let $\{u_d\}_{d\geq 0}$ be a sequence for which there is a polynomial $Q$ of degree $m$ such that $u_d = Q(d)$ whenever $d \geq \bar{d}$ for some $\bar{d}$. Then there exists a polynomial $R$ such that

$$\sum_{d\geq 0} u_d\, z^d = \frac{R(z)}{(1-z)^{m+1}}.$$

*Proof.* We write the polynomial $Q$ as $Q(d) =: \sum_{k=0}^{m} q_k \binom{d+k}{k}$. Then, for the generating function of the sequence $\{u_d\}_{d\geq 0}$, we have

$$\begin{aligned}\sum_{d\geq 0} u_d\, z^d &= \sum_{d\geq 0} Q(d)\, z^d + \sum_{d\geq 0} (u_d - Q(d))\, z^d = \sum_{k=0}^{m} q_k \sum_{d\geq 0} \binom{d+k}{k} z^d + \sum_{d=0}^{\bar{d}} (u_d - Q(d))\, z^d \\ &= \sum_{k=0}^{m} q_k \frac{1}{(1-z)^{k+1}} + \sum_{d=0}^{\bar{d}} (u_d - Q(d))\, z^d.\end{aligned}$$

The latter indeed takes the form $R(z)/(1-z)^{n+1}$ for some polynomial $R$. $\square$

The previous lemma also enables to reprove (4). Indeed, we notice that the lower and upper bounds derived in [3] yield

$$\dim \mathcal{S}_d^r(\Delta_n) = N\binom{d+n}{n} - (r+1)F^{\mathrm{int}}\binom{d+n-1}{n-1} + \mathcal{O}(d^{n-2}).$$

Therefore, for $d$ large enough, the quantity

$$u_d := \dim \mathcal{S}_d^r(\Delta_n) - N\binom{d+n}{n} + (r+1)F^{\mathrm{int}}\binom{d+n-1}{n-1} \tag{9}$$

reduces to a polynomial of degree $\leq n-2$. The claim implies that, for some polynomial $R$,

$$\frac{P_{\Delta_n,r}(z)}{(1-z)^{n+1}} - \frac{N}{(1-z)^{n+1}} + \frac{(r+1)F^{\mathrm{int}}}{(1-z)^n} = \sum_{d\geq 0} u_d\, z^d = \frac{R(z)}{(1-z)^{n-1}}.$$

Rearranging the latter, we obtain

$$P_{\Delta_n,r}(z) = N - (r+1)(1-z)F^{\mathrm{int}} + (1-z)^2 R(z),$$

which in turn shows that $P_{\Delta_n,r}(1) = N$ and $P'_{\Delta_n,r}(1) = (r+1)F^{\mathrm{int}}$.

## 3 Application of the method to specific partitions

In this section, we demonstrate the usefulness of our computational method on several specific partitions. We recall that our method relies on the computation of $\dim \mathcal{S}_d^r(\Delta_n)$ for fixed $d$, $r$, and $\Delta_n$. This step was performed using the interactive applet [1] for $n=3$, and other codes in Java and Fortran for $n>3$, all written by Peter Alfeld.

**Alfeld split of a simplex.** We recall that the split of a simplex $A_n$ in $\mathbb{R}^n$ with $\binom{n+1}{k}$ interior $k$-dimensional faces, $0 \leq k \leq n$, is the Alfeld split of $A_n$. For $n=2$, Theorem 9.3 in [9] yields

$$\dim \mathcal{S}_d^r(A_2) = \binom{d+2}{2} + \binom{d+1-\mu}{2} + \binom{d+1-\nu}{2}, \quad \mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor, \quad \nu := r + \left\lceil \frac{r+1}{2} \right\rceil.$$

For $n=3$ and $r=0,1,2,3$, we were able to compute enough values of the dimensions to derive the sequence $\mathbf{a}^{(r)} := (a_0^{(r)}, a_1^{(r)}, a_2^{(r)}, \ldots)$ of coefficients of the polynomial $P_{A_3,r}$ with certainty. We obtained

$$\begin{aligned}
\mathbf{a}^{(0)} &= (1,1,1,1,0,\ldots)\\
\mathbf{a}^{(1)} &= (1,0,0,0,3,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(2)} &= (1,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(3)} &= (1,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots).
\end{aligned}$$

For $r \geq 4$, the dimensions we could compute yielded the start of the sequence $\mathbf{a}^{(r)}$ with certainty, but we cannot be totally sure that all nonzero coefficients have been found. We obtained

$$\begin{aligned}
\mathbf{a}^{(4)} &= (1,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(5)} &= (1,0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(6)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(7)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,\ldots)\\
\mathbf{a}^{(8)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,\ldots).
\end{aligned}$$

Inspection of the sequences $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(8)}$ strongly suggests the pattern of nonzero coefficients

$$a^{(r)}_{1+2r} = a^{(r)}_{2+2r} = a^{(r)}_{3+2r} = 1 \quad \text{for } r \text{ even}, \qquad a^{(r)}_{2+2r} = 3 \quad \text{for } r \text{ odd}.$$

For $n = 4, 5, 6$, we were also able to compute some values of the dimensions for the space of $\mathcal{C}^r$-splines of degree $\leq d$ over the Alfeld split $A_n$. These investigations lead us to the following conjecture:

$$\text{(10)} \quad \dim \mathcal{S}^r_d(A_n) = \binom{d+n}{n} + \begin{cases} n\dbinom{d+n-\frac{r+1}{2}(n+1)}{n}, & \text{if } r \text{ is odd},\\[2ex] \dbinom{d+n-1-\frac{r}{2}(n+1)}{n} + \cdots + \dbinom{d-\frac{r}{2}(n+1)}{n}, & \text{if } r \text{ is even}. \end{cases}$$

Let us note that for even values of $r$ the formula can be expressed differently since

$$\sum_{j=1}^{n} \binom{d+n-\frac{r}{2}(n+1)-j}{n} = \binom{d+n-\frac{r}{2}(n+1)}{n+1} - \binom{d-\frac{r}{2}(n+1)}{n+1}.$$

The result can be equivalently formulated via the polynomial $P_{A_n,r}$ of (3) as

$$P_{A_n,r}(z) = \begin{cases} 1 + n\, z^{\frac{r+1}{2}(n+1)}, & r \text{ odd},\\[2ex] 1 + \sum\limits_{j=1}^{n} z^{\frac{r}{2}(n+1)+j}, & r \text{ even}. \end{cases}$$

We report below further conjectures produced from our method. Descriptions and illustrations of the corresponding tetrahedral partitions are available in Alfeld's applet menu, see [1].

**Type-I split of a cube ($B_I$).** This partition of a cube consists of six tetrahedra, all sharing one main diagonal of the cube. This diagonal is the only interior edge of the partition. There are no interior split points. Type-I split has 6 interior triangular faces, and 18 boundary edges comprised of 12 edges of the cube and six diagonals of its faces. Based on computations for $r \leq 8$, we conjecture that

$$\dim \mathcal{S}_d^r(B_I) = \binom{d+3}{3} + 3\binom{d+3-(r+1)}{3} + \begin{cases} 2\binom{d+3-\frac{3r+3}{2}}{3}, & r \text{ odd}, \\ \binom{d+3-\frac{3r+2}{2}}{3} + \binom{d+3-\frac{3r+4}{2}}{3}, & r \text{ even}. \end{cases}$$

**Worsey–Farin split of a tetrahedron ($WF$).** This partition is a refinement of the Alfeld split $A_3$ of a tetrahedron obtained by applying the Clough–Tocher split $A_2$ to each face of the tetrahedron. The Worsey–Farin split consists of 12 subtetrahedra meeting at one interior point. This partition has 18 interior triangular faces and 8 interior edges. Based on computations for $r \leq 8$, we conjecture that

$$\dim \mathcal{S}_d^r(WF) = \binom{d+3}{3} + \begin{cases} 8\binom{d+3-\frac{3r+3}{2}}{2} \\ 4\binom{d+3-\frac{3r+2}{2}}{3} + 4\binom{d+3-\frac{3r+4}{2}}{3} \end{cases}$$

$$+ \begin{cases} 3\binom{d+3-(2r+2)}{3}, & r \text{ odd}, \\ \binom{d+3-(2r+1)}{3} + \binom{d+3-(2r+2)}{3} + \binom{d+3-(2r+3)}{3}, & r \text{ even}. \end{cases}$$

**Generic octahedron ($OCT$).** This partition of an octahedron consists of eight tetrahedra meeting at one interior split point. This split point cannot be collinear with any two vertices of the octahedron. There are 12 interior triangular faces and 6 interior edges in this partition. Based on computations for $r \leq 8$, we conjecture that

$$\dim \mathcal{S}_d^r(OCT) = \binom{d+3}{3}$$

$$+ \begin{cases} (r+1)\binom{d+3-(2r+1)}{3} + 7\binom{d+3-(2r+2)}{3} - (r+1)\binom{d+3-(2r+3)}{3}, & r = 2 \bmod 3, \\ (r+3)\binom{d+3-(2r+1)}{3} + 3\binom{d+3-(2r+2)}{3} - (r-1)\binom{d+3-(2r+3)}{3}, & \text{otherwise}. \end{cases}$$

**Generic 8-cell ($C_8$).** The easiest way to visualize this partition is to start with a refinement of the Alfeld split $A_3$ of a tetrahedron obtained by applying the Clough–Tocher split $A_2$ to two faces of the tetrahedron. Let us denote the new split points on the face $u$ and $v$. This partition consists of 8 subtetrahedra meeting at one interior point. Note that the vertices $u$ and $v$ can be moved to the exterior of the original tetrahedron without changing the topology of the partition. This process results in a partition that has the same number of interior and boundary faces, edges, and vertices as the octahedral partition described above. However, connectivities of the faces are different. For example, each interior edge of the octahedral split is shared by exactly four tetrahedra. In the 8-cell, two interior edges are shared by five tetrahedra, another two are shared by four tetrahedra, and the remaining two edges are shared by three tetrahedra. Based on computations for $r \leq 8$, we conjecture that, for $r \geq 2$,

$$\dim \mathcal{S}_d^r(C_8) = \binom{d+3}{3}$$

$$+ \begin{cases} 2r\binom{d+3-(2r+1)}{3} - (2r-9)\binom{d+3-(2r+2)}{3} - 2\binom{d+3-(2r+3)}{3}, & r \text{ odd}, \\ (2r+1)\binom{d+3-(2r+1)}{3} - (2r-7)\binom{d+3-(2r+2)}{3} - \binom{d+3-(2r+3)}{3}, & r \text{ even}. \end{cases}$$

We note that the cases $r = 0$ and $r = 1$ do not follow the general pattern.

# 4 Discussion

**Towards theoretical improvements.** The main shortcomings of our method are its high complexity and limited reliability.

*Complexity.* At present, we need to compute an exponential in $n$ number of values of spline dimensions. As $n$ increases, the cost of computing each dimension goes up. This quickly becomes prohibitive. One way to resolve this issue is to lower the bound on $d^\star$. Ideally, it would be a drop from $r2^n + 1$ down to a quantity that is linear in $n$. Such estimate on the lower bound on $d^\star$ is supported by several observations. If $n = 2$, for shellable triangulations, we have $d^\star \leq 3r + 2$. When $n = 3$, reasonably low values of $d^\star$ can be inferred for the examples of Section 3. We also observed linear behavior in $n$ of $d^\star$ for the Alfled splits. One can also envision that further theoretical information will help to reduce the number of computations. For instance, if a specific partition is known to yield nonnegative coefficients $a_k$, then we can stop computing $\dim \mathcal{S}_d^r(\Delta_n)$ as soon as the conditions $\sum_{k=0}^d a_k = F_n$ and $\sum_{k=0}^d k a_k = (r+1)F_{n-1}^{\text{int}}$ are satisfied.

*Reliability.* Even if all necessary values of $\dim \mathcal{S}_d^r(\Delta_n)$ are available for a fixed $r$, the formula we deduce is only valid for this fixed $r$. At present, the formula we infer for all values of $r$ relies on a plausible guess. Some theoretical information on the type of dependence of $\dim \mathcal{S}_d^r(\Delta_n)$ on $r$ would be decisive in this respect. The results of Section 3 suggest dependence on the parity of $r$, sometimes dependence on divisibility of $r$ by $3$, and occasionally the predicted dependence is not valid for smaller values of $r$.

**Towards computational improvements.** To compute the dimension of $\mathcal{S}_d^r(\Delta_n)$, Alfeld's codes translate the set of smoothness conditions into a linear system for the Bernstein–Bézier coefficients, then the matrix of the system is reduced by Gaussian elimination, and its rank is determined. It may be possible to find faster alternatives. The discussion in [7] hints at a practical method using Gröbner bases. Additionally, when computing $\dim \mathcal{S}_d^r(\Delta_n)$, it should be possible to use the knowledge of the dimensions of the spaces with lower degree and smoothness, since the values $\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \leq d \leq d^\star + n\}$ are determined sequentially. Finally, to deduce the coefficients $a_k$, it may be sensible to compute only the quantities $\delta_d^r(\Delta_n)$ appearing in (8), or some suitable linear combinations of $\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \leq d \leq d^\star + n\}$. This latter approach could take advantage of the fact that the sequence $\{a_k\}$ appears to have only few nonzero terms.

**A optimistic final perspective.** Should the theoretical and computational improvements materialize, a stand-alone program for the explicit determination of the dimensions ought to be implemented. With modern (or future) computational power, the dimension formulas could be obtained for a wide variety of partitions. It is not unrealistic that some expressions for the coefficients $a_k$ could then be inferred in terms of the smoothness $r$, the combinatorial parameters, and other topological parameters — especially in the generic case where the geometry does not play a role.

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