Concave Mirsky Inequality and Low-Rank Recovery

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Abstract

We propose a simple proof of a generalized Mirsky inequality comparing the differences of singular values of two matrices with the singular values of their difference. We then discuss the implication of this generalized inequality for the recovery of low-rank matrices via concave minimization.

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In Compressive Sensing, the success of sparse recovery via ℓ_1-minimization is characterized by the so-called null space property (see e.g. [4, Section 4.1]). There is a natural extension of this characterization to low-rank recovery via Schatten 1-norm (aka nuclear) norm minimization. The argument relies on the classical Mirsky inequality, which states that

$$\sum_{i=1}^{n} |\sigma_i(X) - \sigma_i(Y)| \leq \sum_{i=1}^{n} \sigma_i(X - Y) \qquad \text{for all } X, Y \in \mathbb{C}^{n \times n},$$

where $\sigma_1(M), \ldots, \sigma_n(M)$ denote the singular values of a matrix $M \in \mathbb{C}^{n\times n}$ arranged in non-increasing order. When considering low-rank recovery via the intuitively more potent Schatten q-quasinorm minimization for $0 < q < 1$, characterizing its success through an adapted null space property would require the following generalization of Mirsky inequality:

$$\sum_{i=1}^{n} |\sigma_i(X)^q - \sigma_i(Y)^q| \overset{?}{\leq} \sum_{i=1}^{n} \sigma_i(X - Y)^q \qquad \text{for all } X, Y \in \mathbb{C}^{n \times n}.$$

This was conjectured in [5, Section VI]. In fact, a stronger version with the qth power replaced by a concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$ was conjectured by W. Miao and appeared in [2, Conjecture 6]. Some erroneous proofs were then published ([9] suffers from a confusion between singular values and eigenvalues; as for [8], Theorem 2 cannot be true — take e.g. $f(x) = x$

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and $M = 0$: no matter how M_π is defined, summing the inequalities obtained with $N = I$ and with $N = -I$ yields $2nt \le \mathcal{O}(t^2)$ for $t > 0$, which is impossible). K. Audenaert [1] provided a (so far unpublished) proof based on intricate applications of f-versions of Thompson–Freede inequalities [7]. The purpose of this short note is to offer a simpler proof of the f-version of Mirsky inequality[1]. A weakening of the result with 'the absolute value outside the sum' would be an easy consequence of Bourin–Uchiyama triangle inequality [3, Corollary 2.6], which stipulates that, for any $A, B \in \mathbb{C}^{n\times n}$, the moduli of A, B, and $A+B$ (i.e., the positive semidefinite matrices in their polar decompositions) satisfy

$$f(|A+B|) \preceq U f(|A|) U^* + V f(|B|) V^* \qquad \text{for some unitary matrices } U, V \in \mathbb{C}^{n\times n}.$$

Here is the result formally stated for rectangular matrices with 'the absolute value inside the sum'.

Theorem 1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function satisfying $f(0) = 0$. Then

$$(1) \qquad \sum_{i=1}^{\min\{n_1,n_2\}} |f(\sigma_i(X)) - f(\sigma_i(Y))| \le \sum_{i=1}^{\min\{n_1,n_2\}} f(\sigma_i(X-Y)) \qquad \text{for all } X, Y \in \mathbb{C}^{n_1\times n_2}.$$

Proof. We only have to consider the case of square matrices, i.e., $n_1 = n_2 =: n$. Indeed, if $n_1 > n_2$, say, applying the result for square matrices to $\widetilde{X} = \left[\, X \,\middle|\, 0 \,\right] \in \mathbb{C}^{n_1\times n_1}$ and $\widetilde{Y} = \left[\, Y \,\middle|\, 0 \,\right] \in \mathbb{C}^{n_1\times n_1}$ automatically gives (1). The argument now consists of three steps: reduction to the case where f is a hook function and $X - Y$ has rank one, reduction to an inequality involving no matrices (the tools put to use being the original Mirsky inequality and a result of Thompson about rank-one perturbation for singular values), and the justification of the inequality.

Step 1a: We first claim that it is (necessary and) sufficient to prove

$$(2) \qquad \sum_{i=1}^{n} |\min\{1, \sigma_i(X)\} - \min\{1, \sigma_i(Y)\}| \overset{?}{\le} \sum_{i=1}^{n} \min\{1, \sigma_i(X-Y)\} \qquad \text{for all } X, Y \in \mathbb{C}^{n\times n}.$$

This is the idea of [1]. The justification is that any concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$ can be uniformly approximated on $[0, \mu]$, $\mu := \max\{\sigma_1(X), \sigma_1(Y), \sigma_1(X-Y)\}$, by a positive linear combination of hook functions $h_t(x) := \min\{t, x\}$, $t > 0$. Indeed, for any integer $m \ge 1$, we may consider points $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = \mu$ such that $f(t_k) = k f(\mu)/m$ for $k = 0, \ldots, m$. Since the function f is necessarily nondecreasing, it is uniformly approximated on $[0, \mu]$ with error at most $f(\mu)/m$ by the piecewise linear function passing through the points $(t_0, f(t_0)), (t_1, f(t_1), \ldots, (t_m, f(t_m))$. By concavity of f, the slopes $\alpha_1, \ldots, \alpha_m$ of the pieces are nonincreasing, which allows us to write the piecewise linear approximant as $\sum_{k=1}^m \beta_k \min\{t_k, x\}$ with $\beta_k = \alpha_k - \alpha_{k+1} \ge 0$. Thus, it is (necessary and) sufficient to prove (1) for the functions h_t, $t > 0$. Finally, considering tX and tY instead of X and Y, we see that it is (necessary and) sufficient to prove (1) for $h(x) := \min\{1, x\}$ in the case $n_1 = n_2 = n$.

[1] To be clear, the inequality provided in [1] is stronger, as it deals with incomplete sums, too, whereas our approach has the advantage of simplicity while being sufficient for the purpose of low-rank recovery.

Step 1b: We next claim that it is (necessary and) sufficient to prove

$$(3) \quad \sum_{i=1}^{n} |\min\{1,\sigma_i(X)\} - \min\{1,\sigma_i(Y)\}| \overset{?}{\leq} \min\{1,\sigma_1(X-Y)\} \qquad \text{whenever } \operatorname{rank}(X-Y)=1.$$

Indeed, in general, let us consider the singular value decomposition $X - Y = \sum_{j=1}^{n} s_j u_j v_j^*$. Then, setting $Z_\ell := Y + \sum_{j=1}^{\ell} s_j u_j v_j^*$ for $\ell = 0, \ldots, n$, we see $Z_\ell - Z_{\ell-1} = s_\ell u_\ell v_\ell^*$ has rank one, hence

$$\begin{aligned}
\sum_{i=1}^{n} |h(\sigma_i(X)) - h(\sigma_i(Y))| &= \sum_{i=1}^{n} |h(\sigma_i(Z_n)) - h(\sigma_i(Z_0))| \leq \sum_{i=1}^{n}\sum_{\ell=1}^{n} |h(\sigma_i(Z_\ell)) - h(\sigma_i(Z_{\ell-1}))| \\
&= \sum_{\ell=1}^{n}\sum_{i=1}^{n} |h(\sigma_i(Z_\ell)) - h(\sigma_i(Z_{\ell-1}))| \underset{(3)}{\leq} \sum_{\ell=1}^{n} h(\sigma_1(Z_\ell - Z_{\ell-1})) \\
&= \sum_{\ell=1}^{n} h(s_\ell) = \sum_{\ell=1}^{n} h(\sigma_\ell(X-Y)).
\end{aligned}$$

Step 2: We now claim that it is (necessary and) sufficient to prove that, for any $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$ with $\alpha_{i+1} \leq \beta_i$ and $\beta_{i+1} \leq \alpha_i$,

$$(4) \qquad \sum_{i=1}^{n} |\min\{1,\alpha_i\} - \min\{1,\beta_i\}| \overset{?}{\leq} 1.$$

Indeed, the fact that the hook function h is 1-Lipschitz and the original Mirsky inequality yield

$$\sum_{i=1}^{n} |\min\{1,\sigma_i(X)\} - \min\{1,\sigma_i(Y)\}| \leq \sum_{i=1}^{n} |\sigma_i(X) - \sigma_i(Y)| \leq \sum_{i=1}^{n} \sigma_i(X-Y),$$

so it is (necessary and) sufficient to prove that

$$\sum_{i=1}^{n} |\min\{1,\sigma_i(X)\} - \min\{1,\sigma_i(Y)\}| \overset{?}{\leq} 1 \qquad \text{whenever } \operatorname{rank}(X-Y)=1.$$

But Thompson showed in [6, Theorem 1] that, if $X - Y$ has rank one, then $\alpha_i := \sigma_i(X)$ and $\beta_i := \sigma_i(Y)$ satisfy $\alpha_{i+1} \leq \beta_i$ and $\beta_{i+1} \leq \alpha_i$. This explains the sufficiency of (4). It is necessary, too, since Thompson also showed that any $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$ with $\alpha_{i+1} \leq \beta_i$ and $\beta_{i+1} \leq \alpha_i$ can be realized as singular values of two matrices whose difference has rank one.

Step 3: It remains to justify (4). To this end, we set $\alpha_i' := \min\{1,\alpha_i\}$ and $\beta_i' := \min\{1,\beta_i\}$, so that $\alpha_{i+1}' \leq \alpha_i'$, $\beta_{i+1}' \leq \beta_i'$, $\alpha_{i+1}' \leq \beta_i'$, and $\beta_{i+1}' \leq \alpha_i'$. Observing that $\min\{\alpha_i',\beta_i'\} \geq \max\{\alpha_{i+1}',\beta_{i+1}'\}$ (with the understanding that $\alpha_{n+1}' = \beta_{n+1}' = 0$), we derive that

$$\begin{aligned}
\sum_{i=1}^{n} |\min\{1,\alpha_i\} - \min\{1,\beta_i\}| &= \sum_{i=1}^{n} \left[\max\{\alpha_i',\beta_i'\} - \min\{\alpha_i',\beta_i'\}\right] \\
&\leq \sum_{i=1}^{n} \left[\max\{\alpha_i',\beta_i'\} - \max\{\alpha_{i+1}',\beta_{i+1}'\}\right] = \max\{\alpha_1',\beta_1'\} \leq 1.
\end{aligned}$$

This establishes (4). The proof is therefore complete. □

We now discuss some implications of the concave Mirsky inequality to low-rank recovery. The first observation we put forward is a null space characterization for the success of low-rank recovery via concave minimization. The proof is very similar to the one given in [4, Theorem 4.40] and is consequently omitted.

Theorem 2. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function satisfying $f(0) = 0$. Given a linear map $\mathcal{A}$ from $\mathbb{C}^{n_1 \times n_2}$ to $\mathbb{C}^m$, every matrix $X \in \mathbb{C}^{n_1 \times n_2}$ of rank at most r acquired as $y = \mathcal{A}(X)$ is the unique solution of

$$\underset{Z \in \mathbb{C}^{n_1 \times n_2}}{\text{minimize}} \sum_{j=1}^{\min\{n_1,n_2\}} f(\sigma_j(Z)) \qquad \text{subject to } \mathcal{A}(Z) = y \tag{5}$$

if and only if

$$\sum_{j=1}^{r} f(\sigma_j(M)) < \sum_{\ell=r+1}^{\min\{n_1,n_2\}} f(\sigma_\ell(M)) \qquad \text{for all } M \in \ker(\mathcal{A}) \setminus \{0\}. \tag{6}$$

We emphasize that the property (6) is not vacuous. In fact, it occurs at least as often as low-rank recovery via nuclear norm minimization is successful, according to the following observation, believed to be formalized here for the first time.

Corollary 3. Success of recovery via nuclear norm minimization ($f = \mathrm{id}$ in (5)) implies success of recovery via the minimization (5) for any concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$.

Proof. We prove that (6) for $f = \mathrm{id}$ implies (6) for any concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$. To this end, we remark that (6) can be rewritten as

$$\sum_{j=1}^{r} \frac{1}{\sum_{\ell=r+1}^{\min\{n_1,n_2\}} f(\sigma_\ell(M))/f(\sigma_j(M))} < 1 \qquad \text{for all } M \in \ker(\mathcal{A}) \setminus \{0\}. \tag{7}$$

Suppose that the latter holds for $f = \mathrm{id}$. To derive that it holds for an arbitrary concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$, we can simply observe that $f(\sigma_\ell)/f(\sigma_j) \geq \sigma_\ell/\sigma_j$ whenever $\sigma_\ell \leq \sigma_j$, which follows from the fact that

$$\sigma_\ell = \left(1 - \frac{\sigma_\ell}{\sigma_j}\right) \cdot 0 + \frac{\sigma_\ell}{\sigma_j} \cdot \sigma_j \qquad \text{yields} \qquad f(\sigma_\ell) \geq \left(1 - \frac{\sigma_\ell}{\sigma_j}\right) \cdot f(0) + \frac{\sigma_\ell}{\sigma_j} \cdot f(\sigma_j) = \frac{\sigma_\ell}{\sigma_j} \cdot f(\sigma_j).$$

Thus, the left-hand side of (7) for f is at most what it is for id, hence is bounded above by 1. □

In a similar spirit, whenever $0 < p < q \leq 1$, one can show that Schatten p-quasinorm recovery is successful as soon as Schatten q-quasinorm recovery is successful, analogously to a well-known result for nonconvex recovery of sparse vectors (see e.g. [4, Theorem 4.10]).

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