Supplement to the article: On Maximal Relative Projection Constants

Simon Foucart* and Lesław Skrzypek†

Abstract

In this supplementary note, we describe algorithms that compute (lower bounds for) the maximal and quasimaximal relative projection constants. The algorithms are operative in both the real and complex settings. Although there is no guarantee that the lower bounds coincide with the true values, we could verify the exactness on cases where true values are known, i.e., in the real situation with $1 \le m < N \le 10$.

Let us recall from Section 2 of the main text — see specifically (6)-(7) and (3)-(4) — that the quasimaximal and maximal relative projection constants with respect to $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ can be expressed as

(1)
$$\mu_{\mathbb{K}}(m,N) := \frac{1}{N} \max \left\{ \sum_{i,j=1}^{N} |UU^*|_{i,j}, U \in \mathbb{K}^{N \times m}, U^*U = I_m \right\}$$
$$= \frac{1}{N} \max \left\{ \sum_{k=1}^{m} \lambda_k^{\downarrow}(A), A \in \mathbb{K}^{N \times N}, A^* = A, A_{i,i} = 1, |A_{i,j}| = 1 \right\}$$

and

(2)
$$\lambda_{\mathbb{K}}(m,N) := \max \left\{ \sum_{i,j=1}^{N} t_i |UU^*|_{i,j} t_j, t \in \mathbb{K}^N, \|t\|_2 = 1, U \in \mathbb{K}^{N \times m}, U^*U = I_m \right\}$$
$$= \max \left\{ \sum_{k=1}^{m} \lambda_k^{\downarrow}(TAT), T = \mathrm{diag}(t), \|t\|_2 = 1, A \in \mathbb{K}^{N \times N}, A^* = A, A_{i,i} = 1, |A_{i,j}| = 1 \right\}.$$

The maxima in (1) and (2) are not easily computable because of the absolute values. But if we can replace $|UU^*|_{i,j}$ by $A_{i,j}(UU^*)_{i,j}$ for some sign $A_{i,j}$, then the computation becomes easier. The idea is to refine our guess for the optimal U and A iteratively (in the optimal situation, $A = \mathrm{sgn}(UU^*)$ and U is obtained from the eigendecomposition of A). Thus, we propose the following two algorithms:

*Texas A&M University.
†University of South Florida.

For the computation of the quasimaximal relative projection constant $\mu_{\mathbb{K}}(m,N)$:
Initially, choose $U_0 \in \mathbb{K}^{N\times m}$ with $U_0^* U_0 = I_m$ and choose $A_0 = \overline{\mathrm{sgn}}(U_0 U_0^*) \in \mathbb{K}^{N\times N}$.

Then iterate the following scheme, based on the eigendecomposition of A_{n-1}:

$$\begin{aligned}
\alpha_n &:= \frac{1}{N}\sum_{k=1}^{m} \lambda_k^{\downarrow}(A_{n-1})\\
U_n &:= \text{matrix of } m \text{ leading orthonormal eigenvectors of } A_{n-1} \text{ (so } U_n^* U_n = I_m)\\
\beta_n &:= \frac{1}{N}\sum_{i,j=1}^{N} |U_n U_n^*|_{i,j}\\
A_n &:= \overline{\mathrm{sgn}}(U_n U_n^*).
\end{aligned}$$

For the computation of the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$:
Initially, choose $U_0 \in \mathbb{K}^{N\times m}$ with $U_0^* U_0 = I_m$, choose $t_0 \in \mathbb{K}^N$ as the leading eigenvector of $|U_n U_n^*|$, $T_0 = \mathrm{diag}(t_0)$, and choose $A_0 = \overline{\mathrm{sgn}}(U_0 U_0^*) \in \mathbb{K}^{N\times N}$.

Then iterate the following scheme, based on the eigendecomposition of A_{n-1}:

$$\begin{aligned}
\gamma_n &:= \sum_{k=1}^{m} \lambda_k^{\downarrow}(T_{n-1}A_{n-1}T_{n-1})\\
U_n &:= \text{matrix of } m \text{ leading orthonormal eigenvectors of } T_{n-1}A_{n-1}T_{n-1} \text{ (so } U_n^* U_n = I_m)\\
\delta_n &:= \lambda_1^{\downarrow}(|U_n U_n^*|)\\
t_n &:= \text{leading eigenvector of } |U_n U_n^*|, \quad T_n = \mathrm{diag}(t_n)\\
A_n &:= \overline{\mathrm{sgn}}(U_n U_n^*).
\end{aligned}$$

It can be shown that the sequences (α_n), (β_n), (γ_n), and (δ_n) are convergent, by virtue of

$$\alpha_n \le \beta_n \le \alpha_{n+1} \le \cdots \le \mu_{\mathbb{K}}(m,N)$$

and of

$$\gamma_n \le \delta_n \le \gamma_{n+1} \le \cdots \le \lambda_{\mathbb{K}}(m,N).$$

When calling `QMaxRelProjCst_Real(m,N,nTest)` or `QMaxRelProjCst_Complex(m,N,nTest)` and `MaxRelProjCst_Real(m,N,nTest)` or `MaxRelProjCst_Complex(m,N,nTest)`, our implementations return the maximal values of β_n and δ_n, respectively, over `nTest` random initializations, with n arbitrarily chosen so that $\beta_n - \alpha_n \le 10^{-7}$.