On the dimension of multivariate spline spaces, especially on Alfeld splits

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Abstract

A computational method to obtain explicit formulas for the dimension of spline spaces over simplicial partitions is described. The method is applied to conjecture the dimension formula for the Alfeld split of an n-simplex and for several other tetrahedral partitions.

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1 Introduction

Throughout the paper, let Δ_n denote a simplicial partition of a polyhedral domain $\Omega \subseteq \mathbb{R}^n$, so that if any two simplices in Δ_n intersect, then their intersection is a facet of Δ_n. For example, Δ_2 is a triangulation of a polygon $\Omega \subseteq \mathbb{R}^2$, Δ_3 is a tetrahedral partition of a polyhedron $\Omega \subseteq \mathbb{R}^3$. Let $\mathcal{P}_{d,n}$ denote the space of polynomials of degree $\leq d$ in n variables, and let

$$\mathcal{S}_d^r(\Delta_n) := \left\{ s \in \mathcal{C}^r(\Omega) : \; s|_T \in \mathcal{P}_{d,n} \text{ for each simplex } T \in \Delta_n \right\}$$

denote the space of $\mathcal{C}^r$ splines of degree $\leq d$ in n variables over Δ_n. Our goal in this paper is to determine the dimension of this space for some particular partitions Δ_n. For arbitrary partitions, determining a closed formula is still a major open problem, even in the bivariate

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case $n = 2$. In this case, it is known [9, p.240] that if Δ_2 is a shellable (regular with no holes) triangulation, then

$$(1) \qquad \dim \mathcal{S}_d^r(\Delta_2) \ge \binom{d+2}{2} + E_I \binom{d+1-r}{2} - V_I \left[\binom{d+2}{2} - \binom{r+2}{2} \right] + \sum_{v \in \mathcal{V}_I} \sigma_v,$$

where E_I is the number of interior edges, V_I is the number of interior vertices, $\mathcal{V}_I$ is the set of interior vertices of Δ_2, and

$$\sigma_v := \sum_{j=1}^{d-r} \max\{r + j + 1 - j m_v, 0\}, \quad m_v := \text{number of different edge slopes meeting at } v.$$

The right-hand side of (1) is the correct expression for the dimension if $d \ge 3r+1$, see [9, p.248 and p.273]. Not much is known for $d \le 3r$, and it is somewhat staggering that the dimensions of $\mathcal{S}_3^1(\Delta_2)$ and of $\mathcal{S}_2^1(\Delta_2)$ remain uncertain in general. Let us point out that the right-hand side of inequality (1) can be rewritten as a linear combination of binomial coefficients — a form that is favored in this paper:

$$\begin{aligned} \dim \mathcal{S}_d^r(\Delta_2) \ge\ & \binom{d+2}{2} + (E_I - 3V_I) \binom{d+1-r}{2} \\ & + V_I \binom{d+1-\mu}{2} + V_I \binom{d+1-\nu}{2} + \Big(3V_I \ - \sum_{v \in \mathcal{V}_I} m_v \Big) \binom{\mu+1-r}{2}, \end{aligned}$$

where

$$\mu := r + \left\lfloor \frac{r+1}{2} \right\rfloor, \quad \nu := r + \left\lceil \frac{r+1}{2} \right\rceil.$$

In the case of a cell C_2 — a triangulation with one interior vertex v — it is known that the lower bound is the correct dimension, namely

$$\dim \mathcal{S}_d^r(C_2) = \binom{d+2}{2} + (E_I - 3) \binom{d+1-r}{2} + \binom{d+1-\mu}{2} + \binom{d+1-\nu}{2} + (3-m) \binom{\mu+1-r}{2},$$

where $m \le E_I$ is the number of different slopes of the E_I interior edges meeting at v. In fact, the formula for the cell is the basis of the argument used to derive (1). This example demonstrates that the dimension depends not only on the combinatorics of Δ_n — number of vertices, edges, and other faces — but also on its exact geometry. Applying this dimension formula to the Clough–Tocher split Δ_{CT} of a triangle, that is a cell with $m = E_I = 3$, we obtain

$$\dim \mathcal{S}_d^r(\Delta_{CT}) = \binom{d+2}{2} + \binom{d+1-\mu}{2} + \binom{d+1-\nu}{2}.$$

The main experimental result of this paper is a dimension formula for the generalization of the Clough–Tocher split to higher dimensions. The split of a tetrahedron with one interior vertex, four interior edges and six interior faces was introduced in [2]. We shall refer to the split of a simplex in $\mathbb{R}^n$ with one interior vertex and $\binom{n+1}{k}$ interior k-dimensional faces, $0 \le k \le n$, as the Alfeld split A_n. The following is our conjecture on the dimension.

Conjecture 1. The dimension of the space of $\mathcal{C}^r$ splines of degree $\leq d$ in n variables over the Alfeld split A_n of a simplex is given by

$$\dim \mathcal{S}_d^r(A_n) \stackrel{?}{=} \binom{d+n}{n} + \begin{cases} n\dbinom{d+n-\frac{r+1}{2}(n+1)}{n}, & \text{if } r \text{ is odd,} \\ \dbinom{d+n-1-\frac{r}{2}(n+1)}{n} + \cdots + \dbinom{d-\frac{r}{2}(n+1)}{n}, & \text{if } r \text{ is even.} \end{cases}$$

This formula was obtained using the computational method that we introduce in Section 2. In Section 3, we describe the steps leading to Conjecture 1. In this section, we also report without details other formulas obtained via this method for several tetrahedral partitions. In Section 4, we discuss the potential of the method. We conclude with spline-based derivations of Hilbert series in the Appendix.

2 The computational method

In this section, we show how to derive an explicit formula for the dimension of $\mathcal{S}_d^r(\Delta_n)$, in the form of a linear combination of binomial coefficients, using computed values of this dimension for a finite number of parameters r and d. The first subsection reveals why the sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ depends only on a finite number of its values, while the second subsection describes how to obtain the required formula.

2.1 Primitive method via Hilbert polynomial

Let us for now fix the number n of variables, the simplicial partition Δ_n, and the smoothness parameter r. It is well-known in Algebraic Geometry that the dimension of $\mathcal{S}_d^r(\Delta_n)$ agrees with a polynomial of degree n in variable d when d is sufficiently large. This polynomial is called the Hilbert polynomial, and it is denoted by $H := H_{\Delta_n,r}$ throughout this paper. We denote by $d^\star := d^\star_{\Delta_n,r}$ the smallest integer such that

$$\dim \mathcal{S}_d^r(\Delta_n) = H(d) \quad \text{for all } d \geq d^\star.$$

The sequence $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ is determined by its first $d^\star + n + 1$ values. Indeed, the terms

$$\{\dim \mathcal{S}_d^r(\Delta_n),\ d^\star \leq d \leq d^\star + n\}$$

define $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq d^\star}$ by interpolation of the Hilbert polynomial, while the values

$$\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \leq d \leq d^\star - 1\}$$

complete the first $d^\star$ terms of the sequence. The Hilbert polynomial $H(d)$ can also be tracked via Bernstein–Bézier methods, which additionally provide some ground for a conjecture on $d^\star$ (see Appendix for details):

$$d^\star_{\Delta_n,r} \overset{?}{\leq} r2^n + 1. \tag{2}$$

This bound is likely to be an overestimation. The examples of Section 3 hint to this, and so does the improved bound $d^\star_{\Delta_2,r} \leq 3r + 2$ valid when $n = 2$ for a shellable triangulation [8]. Reducing the bound would reduce the number of dimension values to be computed. Since splines with degrees not exceeding smoothness are simply polynomials, we have

$$\dim \mathcal{S}^r_d(\Delta_n) = \binom{d+n}{n} \quad \text{for } d \leq r.$$

Thus, assuming (2), only the $r(2^n - 1) + n + 1$ values $\{\dim \mathcal{S}^r_d(\Delta_n),\ r + 1 \leq d \leq r2^n + n + 1\}$ are left to be computed. An additional saving can be made by also dropping the computations for some degrees larger than d, since we have

$$\left[\dim \mathcal{S}^r_d(\Delta_n) = \dim \mathcal{P}_{d,n} = \binom{d+n}{n}\right] \Longrightarrow \left[\dim \mathcal{S}^r_k(\Delta_n) = \dim \mathcal{P}_{k,n} = \binom{k+n}{n} \text{ for } k \leq d\right].$$

2.2 Sophisticated method via Hilbert series

Let n, Δ_n, and r be fixed. Assuming that computing $\dim \mathcal{S}^r_d(\Delta_n)$ is possible for any $d \geq 0$, the previous method gives us access to the whole sequence $\{\dim \mathcal{S}^r_d(\Delta_n)\}_{d\geq 0}$, but it comes short of supplying an explicit formula. The method presented in this subsection rectifies the situation. It relies on another common concept in Algebraic Geometry — the generating function of the sequence $\{\dim \mathcal{S}^r_d(\Delta_n)\}_{d\geq 0}$, which is known as the Hilbert series. According to [6, Theorem 2.8], it satisfies

$$\sum_{d\geq 0} \dim \mathcal{S}^r_d(\Delta_n)\, z^d = \frac{P(z)}{(1-z)^{n+1}}, \tag{3}$$

for some polynomial $P := P_{\Delta_n,r}$ with integer coefficients. Denoting these coefficients by $a_k = a_{k,\Delta_n,r}$, and denoting the degree of P by $k^\star = k^\star_{\Delta_n,r}$, that is,

$$P(z) = \sum_{k=0}^{k^\star} a_k z^k, \qquad a_{k^\star} \neq 0,$$

two further particulars are established in [6, Theorem 4.5]:

$$P(1) = \sum_{k=0}^{k^\star} a_k = N, \qquad P'(1) = \sum_{k=0}^{k^\star} k a_k = (r+1)\, F^{\text{int}}, \tag{4}$$

where N and $F^{\rm int}$ represent the number of simplices and interior facets of Δ_n, respectively. These statements can also be deduced from Bernstein–Bézier methods, as outlined in the Appendix. In the particular case when Δ_n is a single simplex, the space $\mathcal{S}^r_d(\Delta_n)$ is just the space $\mathcal{P}_{d,n}$ of polynomials of degree d in n variables. Then it can be seen that $P=1$ from the identity

(5)
$$\sum_{d\geq 0}\binom{d+n}{n}z^d=\frac{1}{(1-z)^{n+1}}.$$

This identity is clear for $n=0$ and is inductively obtained by successive differentiations with respect to z for $n\geq 1$. While the derivation of the polynomial P from the dimensions $\dim\mathcal{S}^r_d(\Delta_n)$ was straightforward, identity (5) conversely provides an explicit formula for the dimensions $\dim\mathcal{S}^r_d(\Delta_n)$ in terms of the coefficients of P. Indeed, the formula

(6)
$$\dim\mathcal{S}^r_d(\Delta_n)=\sum_{k=0}^{k^\star}a_k\binom{d+n-k}{n}$$

was isolated in [6] and it follows from

$$\sum_{d\geq 0}\dim\mathcal{S}^r_d(\Delta_n)\,z^d=\sum_{k=0}^{k^\star}a_k\frac{z^k}{(1-z)^{n+1}}=\sum_{k=0}^{k^\star}\sum_{d\geq 0}a_k\binom{d+n}{n}z^{d+k}=\sum_{d\geq 0}\sum_{k=0}^{k^\star}a_k\binom{d+n-k}{n}z^d$$

by identifying the coefficients in front of each z^d. Taking into account that

$$\binom{d+n-k}{n}=\begin{cases}\dfrac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}, & \text{if } d\geq k,\\ 0=\dfrac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}, & \text{if } k-n\leq d\leq k-1,\\ 0\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad, & \text{if } d\leq k-n-1,\end{cases}$$

we observe that, for $d\geq k^\star-n$, the dimension of $\mathcal{S}^r_d(\Delta_n)$ agrees with the Hilbert polynomial

$$H(d):=\sum_{k=0}^{k^\star}a_k\frac{(d-k+n)(d-k+n-1)\cdots(d-k+1)}{n!}.$$

Moreover, for $d=k^\star-n-1$, we have

$$H(k^\star-n-1)-\dim\mathcal{S}^r_{k^\star-n-1}(\Delta_n)=a_{k^\star}\left(\frac{(-1)(-2)\cdots(-n)}{n!}-0\right)=(-1)^n a_{k^\star}\neq 0.$$

The definition of $d^\star$ therefore yields $d^\star=k^\star-n$, and consequently, we see that

$$k^\star=d^\star+n.$$

This was intuitively anticipated because the determination of the sequence $\{\dim\mathcal{S}^r_d(\Delta_n)\}_{d\geq 0}$ requires $d^\star+n+1$ pieces of information while the equivalent determination of the polynomial

P requires the $k^\star+1$ pieces of information corresponding to its coefficients. Now we describe a practical way to determine these coefficients from the computed values $\{\dim \mathcal{S}^r_d(\Delta_n)\}_{d=0}^{d^\star+n}$. It is simply based on the observation that

$$
\begin{aligned}
a_k &= \frac{1}{k!}\frac{d^k P(z)}{dz^k}|_{z=0} = \frac{1}{k!}\frac{d^k}{dz^k}\Big((1-z)^{n+1}\sum_{d\geq 0}\dim \mathcal{S}^r_d(\Delta_n)\, z^d\Big)|_{z=0} \\
&= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{d^{k-\ell}}{dz^{k-\ell}}\Big((1-z)^{n+1}\Big)|_{z=0}\,\frac{d^\ell}{dz^\ell}\Big(\sum_{d\geq 0}\dim \mathcal{S}^r_d(\Delta_n)\, z^d\Big)|_{z=0} \\
&= \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{k-\ell}\frac{(n+1)!}{(n+1-k+\ell)!}\,\ell!\dim \mathcal{S}^r_\ell(\Delta_n) \\
&= \sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+1}{k-\ell}\dim \mathcal{S}^r_\ell(\Delta_n). \qquad (7)
\end{aligned}
$$

In particular, the value of $\dim \mathcal{S}^r_0(\Delta_n)=1$ yields $a_0=1$, then the value of $\dim \mathcal{S}^r_1(\Delta_n)$ yields a_1, the values of $\dim \mathcal{S}^r_1(\Delta_n)$ and of $\dim \mathcal{S}^r_2(\Delta_n)$ yield a_2 and so on. This shows that the computation of the coefficients a_k can be performed sequentially, along with the computation of the dimensions $\dim \mathcal{S}^r_k(\Delta_n)$. As long as $\dim \mathcal{S}^r_k(\Delta_n)$ equals $\binom{k+n}{n}$, identity (5) ensures that the coefficients a_k agree with the coefficients of the constant polynomial $P=1$:

$$a_0=1,\quad a_1=0,\quad a_2=0,\quad \cdots,\quad a_{d_\star}=0,$$

where $d_\star$ denotes the largest integer such that $\dim \mathcal{S}^r_{d_\star}(\Delta_n)=\binom{d_\star+n}{n}$. As a matter of fact, applying (7) to a partition consisting of a single simplex, we obtain

$$0=\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+1}{k-\ell}\binom{\ell+n}{n},\quad k\geq 1.$$

We may therefore also express the coefficient a_k as

$$a_k=\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+1}{k-\ell}\delta^r_\ell(\Delta_n),\quad k\geq 1, \qquad (8)$$

where $\delta^r_\ell(\Delta_n)$ is the codimension of the polynomial space $\mathcal{P}_{\ell,n}$ in the spline space $\mathcal{S}^r_\ell(\Delta_n)$, i.e.,

$$\delta^r_\ell(\Delta_n):=\dim \mathcal{S}^r_\ell(\Delta_n)-\binom{\ell+n}{n},$$

which is less costly to compute than the dimension of $\mathcal{S}^r_\ell(\Delta_n)$. We finally note that at most $\min\{n+2,k-d_\star\}$ nonzero terms enter the sum in (8), since the summand is nonzero only when $\ell\geq k-n-1$ and $\ell\geq d_\star+1$.

3 Application to specific partitions

In this section, we demonstrate the usefulness of our computational method on several specific partitions. In the case of the Alfeld split of a tetrahedron in $\mathbb{R}^3$, we explain in details how the formula was conjectured. The details are omitted for the Alfeld split of a simplex in $\mathbb{R}^n$, and for several tetrahedral partitions in $\mathbb{R}^3$ presented at the end of the section. We recall that our method relies on the computation of $\dim \mathcal{S}_d^r(\Delta_n)$ once d, r, and Δ_n have been fixed. This step was performed using the interactive applet [1] for $n = 3$, and other codes in Java and Fortran for $n > 3$, all written by Peter Alfeld.

3.1 Trivariate splines over the Alfeld split of a tetrahedron

When r is fixed, in order to find an explicit formula for the whole sequence $\{\dim \mathcal{S}_d^r(A_3)\}_{d\geq 0}$ of dimensions for the trivariate $\mathcal{C}^r$ splines over the Alfeld split A_3, we only need to compute

$$\{\dim \mathcal{S}_d^r(A_3), \quad 0 \leq d \leq 8r+4\}.$$

For $r = 1, 2, 3$, we were able to compute enough values of the dimensions to derive the sequence $\mathbf{a}^{(r)} := (a_0^{(r)}, a_1^{(r)}, a_2^{(r)}, \ldots)$ of coefficients of $P_{A_3,r}$ with absolute certainty. We obtained

$$\begin{aligned}
\mathbf{a}^{(0)} &= (1,1,1,1,0,\ldots)\\
\mathbf{a}^{(1)} &= (1,0,0,0,3,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(2)} &= (1,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(3)} &= (1,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots).
\end{aligned}$$

For $r \geq 4$, the values of the dimensions that we were able to compute yielded the start of the sequence $\mathbf{a}^{(r)}$ with some certainty, but we could not be completely sure that all nonzero coefficients had been found. We obtained

$$\begin{aligned}
\mathbf{a}^{(4)} &= (1,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(5)} &= (1,0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(6)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,\ldots)\\
\mathbf{a}^{(7)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,\ldots)\\
\mathbf{a}^{(8)} &= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,\ldots).
\end{aligned}$$

Inspection of the sequences $\mathbf{a}^{(1)}$, …, $\mathbf{a}^{(8)}$ strongly suggests the pattern of nonzero coefficients

$$a_{1+2r} = a_{2+2r} = a_{3+2r} = 1 \quad \text{for } r \text{ even}, \qquad a_{2+2r} = 3 \quad \text{for } r \text{ odd}.$$

Note that this is consistent with both identities of (4). Therefore, according to (6), the conjectured formula for the 3-dimensional Alfeld split is

$$\dim \mathcal{S}_d^r(A_3) \stackrel{?}{=} \binom{d+3}{3} + \begin{cases} 3\binom{d+1-2r}{3}, & r \text{ odd}, \\ \binom{d+2-2r}{3} + \binom{d+1-2r}{3} + \binom{d-2r}{4}, & r \text{ even}. \end{cases} \tag{9}$$

We also report the value of the largest integer $d_\star$ for which the spline space $\mathcal{S}_{d_\star}^r(A_3)$ reduces to the polynomial space $\mathcal{P}_{d_\star,3}$ and the value of the smallest integer $d^\star$ for which $\dim \mathcal{S}_{d^\star}^r(A_3)$ agrees with the Hilbert polynomial $H(d^\star)$. They are

$$d_\star = \begin{cases} 2r+1, & r \text{ odd}, \\ 2r, & r \text{ even}, \end{cases} \qquad d^\star = \begin{cases} 2r-1, & r \text{ odd}, \\ 2r, & r \text{ even}. \end{cases}$$

3.2 Multivariate splines over the Alfeld split of a simplex

For $n = 4, 5, 6$, we computed some values of the dimensions for the space of $\mathcal{C}^r$-splines of degree $\leq d$ over the Alfeld split A_n using the codes made available to us by Peter Alfeld. Our computational method suggested the following general formula announced in Conjecture 1

$$\dim \mathcal{S}_d^r(A_n) \stackrel{?}{=} \binom{d+n}{n} + \begin{cases} n\binom{d+n-\frac{r+1}{2}(n+1)}{n}, & \text{if } r \text{ is odd}, \\ \sum_{j=1}^{n} \binom{d+n-\frac{r}{2}(n+1)-j}{n}, & \text{if } r \text{ is even}. \end{cases}$$

Let us note that for even values of r the formula can be expressed differently since

$$\sum_{j=1}^{n} \binom{d+n-\frac{r}{2}(n+1)-j}{n} = \binom{d+n-\frac{r}{2}(n+1)}{n+1} - \binom{d-\frac{r}{2}(n+1)}{n+1}.$$

We also note that the result can be equivalently formulated via the polynomial $P_{A_n,r}$ of (3) as

$$P_{A_n,r}(z) = \begin{cases} 1 + n\, z^{\frac{r+1}{2}(n+1)}, & r \text{ odd}, \\ 1 + \sum_{j=1}^{n} z^{\frac{r}{2}(n+1)+j}, & r \text{ even}. \end{cases} \tag{10}$$

Finally, we point out the values of the integers $d_\star$ and $d^\star$ as

$$d_\star = \begin{cases} \frac{r+1}{2}(n+1)-1, & r \text{ odd}, \\ \frac{r}{2}(n+1), & r \text{ even}, \end{cases} \qquad d^\star = \begin{cases} \frac{r+1}{2}(n+1)-n, & r \text{ odd}, \\ \frac{r}{2}(n+1), & r \text{ even}. \end{cases}$$

3.3 Trivariate splines over some tetrahedral partitions

In this section, we report conjectured formulas for several tetrahedral partitions obtained by our method and comment on the information they bring out. The names of the partitions considered below follow the names in Alfeld's applet menu, see [1] for illustrations.

Type-I split of a cube. This partition of a cube consists of 6 tetrahedra, all sharing one main diagonal of the cube. This diagonal is the only interior edge of the partition. There are no interior split points. Type-I split has 6 interior triangular faces, and 18 boundary edges comprised of twelve edges of the cube and six diagonals of its faces. Based on formula (12) for $r = 0$ and on computations for $r \in \{1, \dots, 8\}$, we conjecture that

$$\dim \mathcal{S}_d^r(\Delta_3) \overset{?}{=} \binom{d+3}{3} + 3\binom{d+3-(r+1)}{3} + \begin{cases} 2\binom{d+3-\frac{3r+3}{2}}{3}, & r \text{ odd}, \\ \binom{d+3-\frac{3r+2}{2}}{3} + \binom{d+3-\frac{3r+4}{2}}{3}, & r \text{ even}, \end{cases}$$

in which case we have

$$d_\star \overset{?}{=} r, \qquad d^\star \overset{?}{=} \begin{cases} \dfrac{3r-3}{2}, & r \text{ odd}, \\ \dfrac{3r-2}{2}, & r \text{ even}. \end{cases}$$

Worsey–Farin split of a tetrahedron. This partition is a refinement of the Alfeld split A_3 of a tetrahedron obtained by applying the Clough–Tocher split A_2 to each face of the tetrahedron. The Worsey–Farin split consists of 12 subtetrahedra meeting at 1 interior point. This partition has 18 interior triangular faces, and 8 interior edges. Based on formula (12) for $r = 0$ and on computations for $r \in \{1, \dots, 8\}$, we conjecture that

$$\dim \mathcal{S}_d^r(\Delta_3) \overset{?}{=} \binom{d+3}{3} + \begin{cases} 8\binom{d+3-\frac{3r+3}{2}}{2} \\ 4\binom{d+3-\frac{3r+2}{2}}{3} + 4\binom{d+3-\frac{3r+4}{2}}{3} \end{cases}$$

$$+ \begin{cases} 3\binom{d+3-(2r+2)}{3}, & r \text{ odd}, \\ \binom{d+3-(2r+1)}{3} + \binom{d+3-(2r+2)}{3} + \binom{d+3-(2r+3)}{3}, & r \text{ even}, \end{cases}$$

in which case we have

$$d_\star \stackrel{?}{=} \begin{cases} \dfrac{3r+1}{2}, & r \text{ odd}, \\ \\ \dfrac{3r}{2}, & r \text{ even}, \end{cases} \qquad d^\star \stackrel{?}{=} \begin{cases} 2r-1, & r \text{ odd}, \\ 2r, & r \text{ even}. \end{cases}$$

Generic octahedron. This partition of an octahedron consists of 8 tetrahedra meeting at 1 interior split point which cannot be collinear with any two vertices of the octahedron. There are 12 interior triangular faces and 6 interior edges in this partition. Based on formula (12) for $r = 0$ and on computations for $r \in \{1, \ldots, 8\}$, we conjecture that

$$\dim \mathcal{S}_d^r(\Delta_3) \stackrel{?}{=} \binom{d+3}{3}$$

$$+ \begin{cases} (r+1)\dbinom{d+3-(2r+1)}{3} + 7\dbinom{d+3-(2r+2)}{3} - (r+1)\dbinom{d+3-(2r+3)}{3}, r = 2 \bmod 3, \\ \\ (r+3)\dbinom{d+3-(2r+1)}{3} + 3\dbinom{d+3-(2r+2)}{3} - (r-1)\dbinom{d+3-(2r+3)}{3}, \text{ otherwise}, \end{cases}$$

in which case we have

$$d_\star \stackrel{?}{=} 2r, \qquad d^\star \stackrel{?}{=} 2r.$$

This example suggests that the parity of r does not always dictates the behavior of $\dim \mathcal{S}_d^r(\Delta_n)$ with respect to r. It also suggests that the coefficients a_k are not necessarily independent of r.

Regular octahedron. This partition of an octahedron consists of 8 tetrahedra meeting at 1 interior split point taken to be the intersection point of all three diagonals of the octahedron. Based on formula (12) for $r = 0$ and on computations for $r \in \{1, \ldots, 6\}$, we conjecture that

$$\dim \mathcal{S}_d^r(\Delta_3) \stackrel{?}{=} \binom{d+3}{3} + 3\binom{d+3-(r+1)}{3} + 3\binom{d+3-(2r+2)}{3} + \binom{d+3-(3r+3)}{3},$$

in which case we have

$$d_\star \stackrel{?}{=} r, \qquad d^\star \stackrel{?}{=} 3r.$$

Since the regular octahedron has the same number of interior and boundary faces, edges, and vertices as the generic octahedron, but an additional geometric requirement is imposed, this example confirms that the dimension depends not only on the combinatorics of the partition but also on its geometry.

Generic 8-cell. The easiest way to visualize this partition is to start with a refinement of the Alfeld split A_3 of a tetrahedron obtained by applying the Clough–Tocher split A_2 to two faces of the tetrahedron. We denote the new split points on these faces by u and v. This partition consists of 8 subtetrahedra meeting at 1 interior point. The vertices u and v are moved to the exterior of the original tetrahedron without changing the topology of the partition. This process results in a partition that has the same number of interior and boundary faces, edges, and vertices as both octahedral partitions described above. However, connectivity of the faces is different. For example, each interior edge of the octahedral split is shared by exactly four tetrahedra. In the 8-cell, two interior edges are shared by five tetrahedra, another two are shared by four tetrahedra, and the remaining two edges are shared by three tetrahedra. Based on computations for $r \in \{2, \ldots, 8\}$, we conjecture that, for $r \geq 2$,

$$\dim \mathcal{S}_d^r(\Delta_3) \stackrel{?}{=} \binom{d+3}{3}$$

$$+ \begin{cases} 2r\binom{d+3-(2r+1)}{3} - (2r-9)\binom{d+3-(2r+2)}{3} - 2\binom{d+3-(2r+3)}{3}, & r \text{ odd}, \\ (2r+1)\binom{d+3-(2r+1)}{3} - (2r-7)\binom{d+3-(2r+2)}{3} - \binom{d+3-(2r+3)}{3}, & r \text{ even}, \end{cases}$$

in which case we would have

$$d_\star \stackrel{?}{=} 2r, \qquad d^\star \stackrel{?}{=} 2r.$$

When compared with both octahedral partitions, this example reveals that topology of the partition also influences dimension. Moreover, the example suggests that the predicted dependence of a_k on r is not always valid for small values of r. Indeed, formula (12) for $r = 0$ gives $\mathbf{a}^{(\mathbf{0})} = (1, 3, 3, 1, 0, \ldots)$ and computations for $r = 1$ gives $\mathbf{a}^{(1)} = (1, 0, 0, 4, 3, 0, \ldots)$, which do not follow the pattern

$$\begin{array}{lllll} a_0 = 1, & a_{2r+1} = 2r, & a_{2r+2} = 9 - 2r, & a_{2r+3} = -2, & r \text{ odd}, \\ a_0 = 1, & a_{2r+1} = 2r+1, & a_{2r+2} = 7 - 2r, & a_{2r+3} = -1, & r \text{ even}. \end{array}$$

4 Discussion

Supersplines. The superspline spaces are obtained by imposing additional smoothness across some faces of a simplicial partition. The arguments given in the Appendix to establish the form of the Hilbert series carry through for certain superspline spaces — the Algebraic Geometry arguments probably do as well. Since Alfeld's trivariate applet can handle supersmoothness, it is possible to conjecture formulas for superspline spaces over specific partitions in the same way as we did in Section 3. This investigation will be informative to quantify the conjecture that smoothness across j-faces automatically implies some higher smoothness across i-faces for $i < j$.

Towards theoretical improvements. The main shortcomings of our method are its high complexity and limited reliability.

Complexity. At present, we need a number of computations of spline dimensions which is exponential in n, and each computation is also very costly even for moderate n. This quickly becomes prohibitive. One way to resolve this issue is to lower the bound on $d^\star$. Ideally, it would be a drop from $r2^n + 1$ down to a quantity that is linear in n. This tentative drop of the lower bound on $d^\star$ is supported by several observations: for $n = 2$ and shellable triangulations, we have $d^\star \le 3r + 2$; for $n = 3$, reasonably low values of $d^\star$ are reported in Section 3; and we observed a linear behavior in n of $d^\star$ for the Alfled splits. One can also envision that further theoretical information will help to reduce the number of computations. For instance, if a specific partition is known to yield nonnegative coefficients a_k, then we can stop computing $\dim \mathcal{S}_d^r(\Delta_n)$ as soon as the conditions $\sum_{k=0}^d a_k = F_n$ and $\sum_{k=0}^d k a_k = (r+1)F_{n-1}^{\mathrm{int}}$ are satisfied.

Reliability. Even if all necessary values of $\dim \mathcal{S}_d^r(\Delta_n)$ are available for a fixed r, the formula we deduce is only valid for this fixed r. At present, the formula we infer for all values of r relies on a plausible guess. Some theoretical information on the type of dependence of $\dim \mathcal{S}_d^r(\Delta_n)$ on r would be decisive in this respect. The results of Section 3 suggest dependence on the parity of r, sometimes dependence on the divisibility of r by 3, and occasionally the predicted dependence is not valid for smaller values of r.

Towards computational improvements. To compute the dimension of $\mathcal{S}_d^r(\Delta_n)$, Alfeld's codes translate the set of smoothness conditions into a linear system for the Bernstein–Bézier coefficients, then the matrix of the system is reduced by Gaussian elimination, and its rank is determined. It may be possible to find faster alternatives. The discussion in [7] hints at a practical method using Gröbner bases. Additionally, when computing $\dim \mathcal{S}_d^r(\Delta_n)$, it should be possible to use the knowledge of the dimensions of the spaces with lower degree and smoothness, since the values $\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \le d \le d^\star + n\}$ are determined sequentially. Finally, to deduce the coefficients a_k, it may be sensible to compute only the quantities $\delta_d^r(\Delta_n)$ appearing in (8), or some suitable linear combinations of $\{\dim \mathcal{S}_d^r(\Delta_n),\ 0 \le d \le d^\star + n\}$. This latter approach could take advantage of the fact that the sequence $\{a_k\}$ appears to have only few nonzero terms.

A optimistic final perspective. Should the theoretical and computational improvements materialize, a stand-alone program for the explicit determination of the dimensions ought to be implemented. With modern (or future) computational power, the dimension formulas could be obtained for a wide variety of partitions. It is not unrealistic that some expressions for the coefficients a_k could then be inferred in terms of the smoothness r, the combinatorial parameters of Δ_n, and other topological parameters — especially in the generic case where the geometry does not play a role.

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Appendix: Hilbert series via Bernstein–Bézier methods

We justify here form (3) of the Hilbert series without resorting to Algebraic Geometry. It was shown in [5] that the dimension of a subspace of $\mathcal{S}_d^r(\Delta_n)$ for $d \geq r2^n + 1$ is a polynomial in d. This subspace has additional (or super) smoothness $r2^{n-j-1}$ across every j-dimensional face of Δ_n. Our starting point here is the widely accepted assumption that $\dim \mathcal{S}_d^r(\Delta_n)$ is also a polynomial in d for $d \geq r2^n + 1$, since in this case one can apply the technique of partitioning the minimal determining set into non-overlapping subsets associated with each face, see [4]. However, unless additional smoothness is imposed, it is not clear that the cardinalities of the subsets are polynomials in d. Next, we make the following simple claim.

Claim 1. Let $\{u_d\}_{d\geq 0}$ be a sequence for which there exists a polynomial Q of degree m such that $u_d = Q(d)$ whenever $d \geq \bar{d}$ for some $\bar{d}$. Then there exists a polynomial R such that

$$\sum_{d\geq 0} u_d\, z^d = \frac{R(z)}{(1-z)^{m+1}}.$$

Proof. We write the polynomial Q as

$$Q(d) =: \sum_{k=0}^{m} q_k \binom{d+k}{k}.$$

Then, for the generating function of the sequence $\{u_d\}_{d\geq 0}$, we have

$$\begin{aligned}\sum_{d\geq 0} u_d\, z^d &= \sum_{d\geq 0} Q(d)\, z^d + \sum_{d\geq 0} (u_d - Q(d))\, z^d = \sum_{k=0}^{m} q_k \sum_{d\geq 0} \binom{d+k}{k} z^d + \sum_{d=0}^{\bar{d}} (u_d - Q(d))\, z^d \\ &= \sum_{k=0}^{m} q_k \frac{1}{(1-z)^{k+1}} + \sum_{d=0}^{\bar{d}} (u_d - Q(d))\, z^d.\end{aligned}$$

The latter indeed takes the form $R(z)/(1-z)^{n+1}$ for some polynomial R. $\square$

Applying Claim 1 to $\{\dim \mathcal{S}_d^r(\Delta_n)\}_{d\geq 0}$ proves identity (3). The fact that the coefficients of $R = P_{\Delta_n,r}$ are integers follows from (7), since the values of $\dim \mathcal{S}_\ell^r(\Delta_n)$ are all integers. We now establish the two identities of (4). To this end, we notice that the lower and upper bounds derived in [3] yield

$$\dim \mathcal{S}_d^r(\Delta_n) = F_n \binom{d+n}{n} - (r+1) F_{n-1}^{\mathrm{int}} \binom{d+n-1}{n-1} + \mathcal{O}(d^{n-2}).$$

Therefore, for d large enough, the quantity

$$u_d := \dim \mathcal{S}_d^r(\Delta_n) - F_n \binom{d+n}{n} + (r+1) F_{n-1}^{\mathrm{int}} \binom{d+n-1}{n-1} \tag{11}$$

reduces to a polynomial of degree $\le n-2$. Applying Claim 1 to $\{u_d\}_{d\ge 0}$ defined by (11), we see that, for some polynomial R,

$$\frac{P_{\Delta_n,r}(z)}{(1-z)^{n+1}} - \frac{F_n}{(1-z)^{n+1}} + \frac{(r+1)F_{n-1}^{\mathrm{int}}}{(1-z)^n} = \sum_{d\ge 0} u_d\, z^d = \frac{R(z)}{(1-z)^{n-1}}.$$

Rearranging the latter, we obtain

$$P_{\Delta_n,r}(z) = F_n - (r+1)(1-z)F_{n-1}^{\mathrm{int}} + (1-z)^2 R(z),$$

which in turn shows that $P_{\Delta_n,r}(1) = F_n$ and $P'_{\Delta_n,r}(1) = (r+1)F_{n-1}^{\mathrm{int}}$.

We conclude the appendix with a formula for $P_{\Delta_n,0}$. Using the Bernstein–Beziér technique of counting domain points (as in e.g. [4]), it is easy to see that the dimension of $\mathcal{C}^0$ splines on Δ_n depends only on the combinatorics of the partition:

$$\dim \mathcal{S}_d^0(\Delta_n) = \begin{cases} 1, & d = 0, \\ \sum_{k=0}^{n} F_{n-k}\binom{d-1}{n-k}, & d \ge 1, \end{cases}$$

where $F_j,\ 0 \le j \le n,$ is the number of j-dimensional faces in Δ_n. Using (3) and (5) we obtain

$$\begin{aligned} \frac{P_{\Delta_n,0}(z)}{(1-z)^{n+1}} &= 1 + \sum_{d\ge 1}\sum_{k=0}^{n} F_{n-k}\binom{d-1}{n-k} z^d = 1 + \sum_{k=0}^{n} F_{n-k} z^{n-k+1} \sum_{d'\ge 0}\binom{d'+n-k}{n-k} z^{d'} \\ &= 1 + \sum_{k=0}^{n} \frac{F_{n-k} z^{n-k+1}}{(1-z)^{n-k+1}}. \end{aligned}$$

Thus, with the convention that $F_{-1} = 1$, we have

$$P_{\Delta_n,0}(z) = \sum_{k=0}^{n+1} F_{n-k} z^{n+1-k}(1-z)^k. \tag{12}$$

Both identities of (4) are now immediate:

$$P_{\Delta_n,0}(1) = F_n, \qquad \text{and} \qquad P'_{\Delta_n,0}(1) = (n+1)F_n - F_{n-1} = F_{n-1}^{\mathrm{int}},$$

where the last equality reflects the fact that every interior facet is shared by two simplices.

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