When is a subspace of ℓ_∞^N isometrically isomorphic to ℓ_∞^n?

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Abstract

It is shown in this note that one can decide whether an n-dimensional subspace of ℓ_∞^N is isometrically isomorphic to ℓ_∞^n by testing a finite number of determinental inequalities. As a byproduct, an elementary proof is provided for the fact that an n-dimensional subspace of ℓ_∞^N with projection constant equal to one must be isometrically isomorphic to ℓ_∞^n.

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Prelude. The purpose of this note is to settle, in a testable manner, the question raised in the title. To arrive at our answer, an n-dimensional subspace V of ℓ_∞^N is better viewed as an m-codimensional subspace of ℓ_∞^N, $N = m + n$, written as $V = \{x \in \mathbb{R}^N : \langle f^1, x\rangle = \cdots = \langle f^m, x\rangle = 0\}$ for some linearly independent $f^1, \ldots, f^m \in \mathbb{R}^N$. In the simplest case $m = 1$, i.e., $V = \{f\}^\perp$, it is known that $V \cong \ell_\infty^{N-1}$ if and only if $\|f\|_1 \le 2\|f\|_\infty$. This is a side-result of the determination by Blatter and Cheney [3], way back in the 70's, of a formula for the projection constant of hyperplanes in ℓ_∞^N—we will discuss projection constant soon. For the next simpler case $m = 2$, an answer was given in [2], namely $V \cong \ell_\infty^{N-2}$ if and only if there exist linearly independent $f, g \in V^\perp$ and distinct indices $k \neq \ell$ such that $\|f\|_1 \le 2|f_k|$ and $\|g\|_1 \le 2|g_\ell|$. The answer, however, does not directly provide a way to test whether V is isometrically isomorphic to ℓ_∞^{N-2}. The instantiation to the case $m = 2$ of our forthcoming result (Theorem 1) does. Precisely, given linearly independent $f, g \in V^\perp$, defining $\Delta^1, \ldots, \Delta^N \in \ell_\infty^N$ by $\Delta^k = f_k g - g_k f$, one has $V \cong \ell_\infty^{N-2}$ if and only if

$$\text{there exist indices } k \neq \ell \ \text{ such that } \ \max\{\|\Delta^k\|_1, \|\Delta^\ell\|_1\} \le 2\,|\Delta^k_\ell| \ \ (= 2\,|\Delta^\ell_k|).$$

Thus, it is only required to test $2\binom{N}{2}$ presumptive inequalities to settle our question. It is important to note that the above condition is intrinsic to the space V, in that it does not depend on the choice of linearly independent linear vectors f and g in $V^\perp$: e.g. if f was replaced by $cf + dg$, $c \neq 0$, then each Δ^k would be replaced by $c\Delta^k$, which would not affect the status of the presumptive inequalities.

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Notation. The entries of a vector $x \in \mathbb{R}^N$ are marked with a subscript, so that $x = [x_1, \ldots, x_N]^\top$. Superscripts are reserved for indexing sequences of vectors. For instance, a basis of the orthogonal complement $V^\perp$ of an m-codimensional space $V \subseteq \mathbb{R}^N$ is written as $(f^1, \ldots, f^m)$. In condensed form, we write

$$F = \left[\begin{array}{c|c|c} f^1 & \cdots & f^m \end{array} \right] \in \mathbb{R}^{N \times m}.$$

Persisting with this convention, for a matrix $A \in \mathbb{R}^{N \times m}$, its entry located at the intersection of the ith row and the jth column is denoted by a_i^j, its jth column is denoted by a^j, and its ith row is denoted by a_i. More generally, the row-submatrix of A indexed by a set $S \subseteq \{1, \ldots, N\}$ is denoted by A_S. As such, for $A, B \in \mathbb{R}^{N \times m}$, Cauchy–Binet formula reads

$$\det(A^\top B) = \sum_{|S|=m} \det(A_S) \det(B_S).$$

Banach-Mazur distances and projection constants. The so-called Banach–Mazur distance between two finite-dimensional normed spaces V and W is defined[1] as

$$d(V, W) = \min\{\|T\| \, \|T^{-1}\| : \; T \text{ is an isomorphism from } V \text{ to } W\} \geq 1.$$

Thus, a tautological answer is the question of the title can be: "when $d(V, \ell_\infty^n) = 1$". Evidently, this is not satisfying because there is no way (of which we are aware) of efficiently computing this Banach–Mazur distance. As for the projection constant of a subspace V of ℓ_∞^N, it is defined[2] as

$$\lambda(V) = \min\{\|P\| : \; P \text{ is a projection from } \ell_\infty^N \text{ onto } V\} \geq 1.$$

It is well known that $\lambda(V) \leq d(V, \ell_\infty^n)$ and here is a sketched argument for completeness: consider a minimizing isomorphism $T : V \to \ell_\infty^n$; by applying Hahn–Banach theorem componentwise, extend it to $\widetilde{T} : \ell_\infty^N \to \ell_\infty^n$ while preserving its norm; then set $P := T^{-1}\widetilde{T} : \ell_\infty^N \to V$, which is a projection onto V (since $P(v) = T^{-1}T(v) = v$ for all $v \in V$) whose norm satisfies $\|P\| \leq \|T^{-1}\| \, \|\widetilde{T}\| = \|T^{-1}\| \, \|T\| = d(V, \ell_\infty^n)$; and conclude with $\lambda(V) \leq \|P\| \leq d(V, \ell_\infty^n)$. As a result, $d(V, \ell_\infty^n) = 1$ implies $\lambda(V) = 1$. Interestingly, it is also known that $\lambda(V) = 1$ conversely implies $d(V, \ell_\infty^n) = 1$, although none of the many proofs of this result[3] are elementary. Our main result (Theorem 1) actually provides an elementary proof of the equivalence $\lambda(V) = 1 \iff d(V, \ell_\infty^n) = 1$, albeit with the restriction that V is (isometrically isomorphic to) a subspace of ℓ_∞^N. Thus, a better answer to our question is: "when $\lambda(V) = 1$". Arguably, this is a satisfying answer because the projection constant of a subspace of ℓ_∞^N can be computed by linear programming (see e.g. [4] for details)...

[1] The finite-dimensionality is not essential—it simply ensures that the infimum is indeed attained.

[2] Strictly speaking, this quantity is the relative projection constant $\lambda(V, \ell_\infty^N)$ of V—we are making implicit use of the familiar fact that relative and absolute projection constants agree for subspaces of ℓ_∞^N, see e.g. [4].

[3] The result brings up a possible quarrel between West and East claiming precedence: it is often attributed to Nachbin [5], although it seems to have been announced earlier by Akilov [1], see the MathSciNet review MR0077897.

except that most optimization solvers do not work in exact arithmetic, so truly testing the equality $\lambda(V) = 1$ could be problematic. In this sense, the answer we give to the question of the title is "more" satisfying—it entails verifying a finite (but possibly large) number of inequalities which can, on the face of it, be handled symbolically.

The main result. Without further ado, our awaited answer to the question "when is a subspace V of ℓ_∞^N isometrically isomorphic to ℓ_∞^n" materializes as item (i) of the theorem below. Its statement involves an intrinsic basis $(h(S)^k, k \in S)$ of $V^\perp$ associated with a set $S \subseteq \{1, \ldots, N\}$ of size $m = \operatorname{codim}(V)$. Although it is constructed by invoking a fixed basis $(f^1, \ldots, f^m)$ of $V^\perp$, note that it is actually independent of this fixed basis. Its defining formula is, for $k \in S$ and $i = 1, \ldots, N$,

$$h(S)_i^k = \frac{\det(F_S[\mathrm{row}_k \leftarrow \mathrm{row}_i])}{\det(F_S)}, \qquad \text{where} \quad F = \begin{bmatrix} f^1 & \Big| & \cdots & \Big| & f^m \end{bmatrix} \in \mathbb{R}^{N \times m}.$$

On the one hand, the fact that the $h(S)^k$, $k \in S$, belong to $V^\perp$ follows from a Laplace expansion with respect to the kth row, yielding

$$h(S)_i^k = \frac{1}{\det(F_S)} \sum_{j=1}^m (-1)^{k+j} f_i^j \det(F_{S \setminus \{k\}}^{[1:m] \setminus \{j\}}) \qquad \text{for all } i = 1, \ldots, N.$$

In the particular case $m = 2$ and $S = \{k, \ell\}$, we observe that $h(S)^k = \Delta^\ell / \Delta_k^\ell$, which leads to the result mentioned in the prelude. On the other hand, the fact that the $h(S)^k$, $k \in S$, are linearly independent follows from

$$h(S)_i^k = \begin{cases} 0 & \text{if } i \in S \text{ is different from } k, \\ 1 & \text{if } i \in S \text{ is identical with } k. \end{cases}$$

As a consequence, any $f \in V^\perp$ is expressed as $f = \sum_{k \in S} f_k h(S)^k$. In matrix form, this can simply be written as the identity $F = H(S) F_S$, to be used later.

Theorem 1. Given an m-codimensional subspace V of ℓ_∞^N, the following statements are equivalent:

(i) there exists an index set S of size m such that $\|h(S)^k\|_1 \le 2$ for all $k \in S$;

(ii) V is isometrically isomorphic to ℓ_∞^n, $n = N - m$, i.e., $d(V, \ell_\infty^n) = 1$;

(iii) the projection constant of V equals one, i.e., $\lambda(V) = 1$.

The justification of these equivalences owes to the lemmas below. Indeed, the implication (i)⇒(ii) follows from Lemma 2, which is relatively straightforward; the implication (ii)⇒(iii) is a consequence of $\lambda(V) \le d(V, \ell_\infty^n)$; and the implication (iii)⇒(i) follows from Lemma 3, which is the centerpiece of this note.

Lemma 2. For any index set S of size m such that $\det(F_S) \neq 0$,

$$d(V, \ell_\infty^n) \leq \max\big\{1, \max_{k \in S} \|h(S)^k\|_1 - 1\big\}.$$

Lemma 3. Let P be (the matrix of) a projection from ℓ_∞^N onto V with $\|P\| = \lambda(V)$. For any index set S of size m such that $\det(F_S) \neq 0$ and $\det(I - P_S^S) \neq 0$,

$$\max_{k \in S} \|h(S)^k\|_1 - 1 \leq 1 + (\lambda(V) - 1)\|(I - P_S^S)^{-1}\|.$$

Proof of Lemma 2. For $v \in V = \{f^1, \ldots, f^m\}^\perp$, the equality $F^\top v = 0$ yields $F_S^\top v_S + F_{S^c}^\top v_{S^c} = 0$, i.e., $v_S = -F_S^{-\top} F_{S^c}^\top v_{S^c}$. This implies that

$$\|v_S\|_\infty \leq \|F_S^{-\top} F_{S^c}^\top\| \, \|v_{S^c}\|_\infty,$$

where the operator norm is transformed into

$$\|F_S^{-\top} F_{S^c}^\top\| = \max_{k \in S} \sum_{i \in S^c} \big|(F_S^{-\top} F_{S^c}^\top)_k^i\big| = \max_{k \in S} \sum_{i \in S^c} \big|(F_{S^c} F_S^{-1})_i^k\big| = \max_{k \in S} \sum_{i \in S^c} \Big|\sum_{j=1}^m (F_{S^c})_i^j (F_S^{-1})_j^k\Big|$$

$$= \max_{k \in S} \sum_{i \in S^c} \left|\sum_{j=1}^m f_i^j \frac{(-1)^{k+j} \det(F_{S \setminus \{k\}}^{[1:m] \setminus \{j\}})}{\det(F_S)}\right| = \max_{k \in S} \sum_{i \in S^c} \big|h(S)_i^k\big| = \max_{k \in S} \|h(S)_{S^c}^k\|_1.$$

It follows that, for any $v \in V$,

$$\|v\|_\infty = \max\{\|v_{S^c}\|_\infty, \|v_S\|_\infty\} \leq \max\Big\{1, \max_{k \in S} \|h(S)_{S^c}^k\|_1\Big\}\|v_{S^c}\|_\infty.$$

Since $\|v_{S^c}\|_\infty \leq \|v\|_\infty$ also holds for any $v \in V$, we deduce that

$$d(V, \ell_\infty^{N-m}) \leq \max\Big\{1, \max_{k \in S} \|h(S)_{S^c}^k\|_1\Big\}.$$

The announced form of the result makes use $\|h(S)_{S^c}^k\|_1 = \|h(S)^k\|_1 - \|h(S)_S^k\|_1 = \|h(S)^k\|_1 - 1$. $\square$

Proof of Lemma 3. Let P be a (minimal) projection from ℓ_∞^N onto V. Since $I - P$ vanishes on $V = \{f^1, \ldots, f^m\}^\perp$, there exist $y^1, \ldots, y^m \in \mathbb{R}^N$ such that $(I - P)x = \sum_{i=1}^m \langle f^i, x\rangle y^i$ for all $x \in \mathbb{R}^N$. Then, in view of $Px \in V$ for all $x \in \mathbb{R}^N$, we have $0 = \langle f^j, Px\rangle = \langle f^j, x\rangle - \sum_{i=1}^m \langle f^i, x\rangle\langle f^j, y^i\rangle$ for all $j = 1, \ldots, m$. This forces $\langle f^j, y^i\rangle = \delta_{i,j}$ for all $i, j = 1, \ldots, m$. All in all, the projection P can be expressed, for any $x \in \mathbb{R}^N$, as

$$Px = x - \sum_{i=1}^m \langle f^i, x\rangle y^i, \qquad \text{where } y^1, \ldots, y^m \in \mathbb{R}^N \text{ satisfy } \langle f^j, y^i\rangle = \delta_{i,j}.$$

In a more condensed matrix form, this reads

$$P = I_N - YF^\top \qquad \text{where } Y \in \mathbb{R}^{N \times m} \text{ satisfies } F^\top Y = I_m.$$

Relatively to another basis $(g^1, \ldots, g^m)$ of $V^\perp$, written as $G = FM$ for some invertible matrix $M \in \mathbb{R}^{m \times m}$, we have

$$P = I_N - ZG^\top \qquad \text{where } Z = YM^{-\top} \in \mathbb{R}^{N \times m} \text{ satisfies } G^\top Z = I_m.$$

In view of $\sum_{|S|=m} \det(F_S)\det(Y_S) = 1$, which stems from Cauchy–Binet formula, we can find an index set S such that not only $\det(F_S) \neq 0$ but also $\det(Y_S) \neq 0$. The former is needed in the definition of the $h(S)^k$, $k \in S$, and the latter will be needed soon. Fixing this index set S from now on, we take $(g^1, \ldots, g^m)$ to be the basis $(h^k, k \in S)$—dropping the dependence on S for ease of notation. The matrices G, Z, and P thus take the form

$$H = \left[\begin{array}{c} I_m \\ \hline H_{S^c} \end{array}\right], \qquad Z = \left[\begin{array}{c} Z_S \\ \hline Z_{S^c} \end{array}\right], \qquad P = I_N - \left[\begin{array}{c|c} Z_S & Z_S H_{S^c}^\top \\ \hline Z_{S^c} & Z_{S^c} H_{S^c}^\top \end{array}\right].$$

From this expression of P, it follows that

$$\begin{aligned} \|P\| = \max_{i=1,\ldots,N} \sum_{j=1}^N |P_i^j| &\geq \max_{i \in S} \left(|1 - Z_i^i| + \sum_{j \in S \setminus \{i\}} |Z_i^j| + \sum_{j \in S^c} |(Z_S H_{S^c}^\top)_i^j| \right) \\ &\geq \max_{i \in S} \left(1 - |Z_i^i| + \sum_{j \in S \setminus \{i\}} |Z_i^j| + \sum_{j \in S^c} |(Z_S H_{S^c}^\top)_i^j| \right). \end{aligned}$$

Therefore, for any $i \in S$, we obtain after some rearrangement

$$\|P\| - 1 + \alpha_i \geq \beta_i, \qquad \text{where } \alpha_i := |Z_i^i| - \sum_{j \in S \setminus \{i\}} |Z_i^j| \quad \text{and} \quad \beta_i := \sum_{j \in S^c} |(Z_S H_{S^c}^\top)_i^j|.$$

For any $c \in \mathbb{R}^S$, we observe on the one hand that

$$\sum_{i \in S} \beta_i |c_i| = \sum_{j \in S^c} \sum_{i \in S} |(Z_S H_{S^c}^\top)_i^j|\,|c_i| \geq \sum_{j \in S^c} \Big| \sum_{i \in S} (H_{S^c} Z_S^\top)_j^i c_i \Big| = \sum_{j \in S^c} \big| (H_{S^c} Z_S^\top c)_j \big|,$$

and on the other hand that

$$\begin{aligned} \sum_{i \in S} \alpha_i |c_i| &= \sum_{i \in S} |Z_i^i|\,|c_i| - \sum_{\substack{i,j \in S \\ i \neq j}} |Z_i^j|\,|c_i| = \sum_{j \in S} |Z_j^j|\,|c_j| - \sum_{\substack{i,j \in S \\ i \neq j}} |Z_i^j|\,|c_i| \\ &= \sum_{j \in S} \Big(|Z_j^j|\,|c_j| - \sum_{i \in S \setminus \{j\}} |Z_i^j|\,|c_i| \Big) \leq \sum_{j \in S} \Big| \sum_{i \in S} Z_i^j c_i \Big| = \sum_{j \in S} \big| (Z_S^\top c)_j \big|. \end{aligned}$$

We consequently derive that, for any $c \in \mathbb{R}^S$,

$$\big(\|P\| - 1 \big) \sum_{i \in S} |c_i| + \sum_{j \in S} \big| (Z_S^\top c)_j \big| \geq \sum_{j \in S^c} \big| (H_{S^c} Z_S^\top c)_j \big|.$$

At this point, we need the specificity of the index set S to ensure that the matrix Z_S is invertible. This holds true thanks to the identity $F = HF_S$, i.e., $H = FM$ with $M = F_S^{-1}$, which implies that

$Z = YM^{-\top} = YF_S^\top$, so $Z_S = Y_S F_S^\top$ is invertible as the product of two invertible matrices. Hence, for any $\ell \in S$, we can make the choice $c = Z_S^{-\top} h_S^\ell$, for which $c_i = (Z_S^{-1})_\ell^i$ and $Z_S^\top c = h_S^\ell = \delta^\ell$, to arrive at

$$\big(\|P\| - 1\big) \sum_{i \in S} \big|(Z_S^{-1})_\ell^i\big| + 1 \geq \sum_{j \in S^c} \big|h_j^\ell\big|.$$

Restoring the dependence on S, we have shown that there exists an index set S (any S such that $\det(F_S) \neq 0$ and $\det(Y_S) \neq 0$ is suitable) such that

$$\max_{\ell \in S} \|h(S)_{S^c}^\ell\|_1 \leq 1 + \big(\|P\| - 1\big) \max_{\ell \in S} \sum_{i \in S} \big|(Z_S^{-1})_\ell^i\big|.$$

Taking into account that $\|P\| = \lambda(V)$ for a minimal projection, recognizing that the last maximum is $\|Z_S^{-1}\|$, and identifying Z_S with $I - P_S^S$, as apparent from the block-representation of P, completes the argument. $\square$

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