Three topics in multivariate spline theory

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Abstract

We examine three topics at the interface between spline theory and algebraic geometry. In the first part, we show how the concept of domain points can be used to give an original explanation of Dehn–Sommerville equations relating the numbers of i-faces of a simplicial polytope in $\mathbb{R}^n$, $i = 0 \ldots, n-1$. In the second part, we echo some joint works with T. Sorokina and with P. Clarke on computational methods that generate formulas for the dimensions of spline spaces $\mathcal{S}_d^r(\Delta_n)$ of degree $\leq d$ and smoothness r over a fixed simplicial partition Δ_n in $\mathbb{R}^n$. It exploits the specific form of the generating function $\sum_{d \geq 0} \dim \mathcal{S}_d^r(\Delta_n) z^d$ — the so-called Hilbert series. In the third and final part, we state that Schumaker's conjecture about bivariate interpolation at subsets of domain points holds up to degree $d = 17$, with extensions to the trivariate and quadrivariate cases. We also reformulate the conjecture in three different ways, especially as a question about a certain bivariate Vandermonde matrix being a P-matrix.

Key words and phrases: domain points, simplicial polytope, Dehn–Sommerville equations, Hilbert polynomial, Hilbert series, generating functions, linear complementarity problem, multivariate Vandermonde matrix.

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This article highlights some interactions between multivariate spline theory and other areas of mathematics. Firstly, we will show that a concept from spline theory is pertinent in polytope geometry by revealing that Dehn–Sommerville equations are nothing but a count of domain points. Secondly, we will show how the concept of Hilbert series, standard in Algebraic Geometry, can be exploited in spline theory to generate dimension formulas for a fixed simplicial partition. Thirdly, we will show how a conjecture emanating in spline theory can be reformulated in a language pleasing to algebraic geometers in the hope of inspiring attacks from new directions.

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1 A novel approach to Dehn–Sommerville equations

Given a polytope P in $\mathbb{R}^n$, let $f_i(P)$ denote the number of i-dimensional faces of P, $i \in [\![0 : n-1]\!]$. Euler–Poincaré equation states that

$$\sum_{i=0}^{n-1} (-1)^i f_i(P) = 1 + (-1)^{n-1}. \tag{1}$$

In the case of a simplicial polytope S in $\mathbb{R}^n$, this relation is complemented by the Dehn–Sommerville equations, namely

$$\sum_{i=k}^{n-1} (-1)^i \binom{i+1}{k+1} f_i(S) = (-1)^{n-1} f_k(S), \qquad k \in [\![0 : n-1]\!]. \tag{2}$$

Euler–Poincaré equation can be incorporated in the above formulation for $k = -1$ provided the convention $f_{-1} = 1$ is used. Classical justification of (2) rely on shelling [20, page 252], on Euler–Poincaré equation applied to each face of the dual polytope [8, page 105], and on incidence equations with a variant of Euler–Poincaré equation [12, page 146]. The purpose of this section is to reveal a connection with multivariate spline theory that leads to an elementary proof of Dehn–Sommerville equations. It shares a similarity with the proof of [12] by making an indirect use of the generalization

$$\sum_{i=0}^{n-1} (-1)^i f_i(P, F) = (-1)^{n-1} \tag{3}$$

of the Euler–Poincaré equation in which F is an arbitrary face of a polytope P in $\mathbb{R}^n$ and $f_i(P, F)$ denote the number of i-dimensional faces of P containing F, see [12, Theorem 8.3.1]. Spline theory brings to the table the concept of domain points of order d relative to an n-simplex. These are points in the given simplex for which a n-variate polynomial of degree at most d is uniquely determined by its values at these points. Thus, there are $\binom{d+n}{n}$ domain points of order d relative to an n-simplex. If k facets of the n-simplex are removed, then only $\binom{d-k+n}{n}$ domain points of order d are left. In particular, the number of interior domain points is $\binom{d-1}{n}$. Figure 1 gives a 'definition-by-illustration' of the concept of domain points.

1.1 The new proof

For simplicity, we write f_i instead of $f_i(S)$ for the number of i-dimensional faces of the simplicial polytope S. For $d \geq 1$, we equip each facet of S with its domain points of order d. Dehn–Sommerville equations simply follow from counting the domain points in two different ways.[1]

[1] In fact, the equations (2) are equivalent to this twofold count because the arguments are completely reversible.

Figure 1: Left: domain points of order $d = 6$ relative to a triangle (i.e., a 2-simplex); Right: domain points of order $d = 5$ relative to a tetrahedron (i.e., a 3-simplex).

Lemma 1. For $d \geq 1$, the total number of domain points of order d is

$$u_d = \sum_{k=0}^{n-1} f_k \binom{d-1}{k} = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \binom{d+n-\ell}{n-\ell}.$$

Proof. In the first viewpoint, we count the domain points located on the f_0 vertices (i.e., 1 for each vertex), then the remaining domain points located on the f_1 edges (i.e., $d-1$ for each edge), then the remaining domain points located on the f_2 2-faces (i.e., $\binom{d-1}{2}$ for each 2-face), etc. This gives the first expression for u_d.
The second viewpoint intuitively amounts to counting $\binom{d+n-1}{n-1}$ domain points for each of the facets, to which $\binom{d+n-2}{n-2}$ domain points for each of the $(n-2)$-faces are subtracted because they have been counted already, to which $\binom{d+n-3}{n-3}$ domain points for each of the $(n-3)$-faces are added because they have been removed too many times, etc. Although the reasoning is not convincing as such, it is formalized by introducing the smallest face F_ξ containing ξ for each domain point ξ, and then noticing that

$$\sum_{\ell=1}^{n} (-1)^{\ell-1} \sum_{F \text{ is a } (n-\ell)\text{-face of } P} \mathbf{1}_F(\xi) = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell}(P, F_\xi) = 1.$$

The last equality was due to (3). Summing over all ξ gives the second expression for u_d. □

The Dehn–Sommerville equations are now easily obtained by looking at the generating function $G(z) := \sum_{d \geq 1} u_d z^d$. We shall use several times the formula

$$\sum_{j \geq 0} \binom{j+m}{m} z^j = \frac{1}{(1-z)^{m+1}},$$

which is clear for $m = 0$ and is otherwise deduced from the case $m = 0$ by taking the mth derivative with respect to z. On the one hand, we compute

$$
\begin{aligned}
G(z) &= \sum_{d\ge 1}\sum_{k=0}^{n-1} f_k \binom{d-1}{k} z^d = \sum_{k=0}^{n-1} f_k \sum_{d\ge k+1} \binom{d-1}{k} z^d = \sum_{k=0}^{n-1} f_k z^{k+1} \sum_{j\ge 0} \binom{j+k}{k} z^j \\
&= \sum_{k=0}^{n-1} f_k \frac{z^{k+1}}{(1-z)^{k+1}}. \qquad (4)
\end{aligned}
$$

On the other hand, we compute

$$
\begin{aligned}
G(z) &= \sum_{d\ge 1}\sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \binom{d+n-\ell}{n-\ell} z^d = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \sum_{d\ge 1} \binom{d+n-\ell}{n-\ell} z^d \\
&= \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \left(\frac{1}{(1-z)^{n-\ell+1}} - 1 \right) = \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} \frac{1}{(1-z)^{n-\ell+1}} - (1+(-1)^{n-1}) f_{-1}, \qquad (5)
\end{aligned}
$$

where the last step relied on (1) together with the convention $f_{-1} = 1$. Equating the expressions (4) and (5) for $G(z)$ and rearranging yields

$$
\sum_{k=-1}^{n-1} f_k \frac{z^{k+1}}{(1-z)^{k+1}} = \sum_{\ell=1}^{n+1} (-1)^{\ell-1} f_{n-\ell} \frac{1}{(1-z)^{n+1-\ell}}. \qquad (6)
$$

Setting $x := z/(z-1)$, so that $x - 1 = 1/(z-1)$, the latter reads

$$
\sum_{k=-1}^{n-1} (-1)^{k+1} f_k x^{k+1} = \sum_{\ell=1}^{n+1} (-1)^n f_{n-\ell} (x-1)^{n+1-\ell} = (-1)^n \sum_{i=-1}^{n-1} f_i (x-1)^{i+1}.
$$

Identifying the coefficient of x^{k+1} for each $k \in [\![0 : n-1]\!]$ gives

$$
(-1)^{k+1} f_k = (-1)^n \sum_{i=k}^{n-1} f_i \binom{i+1}{k+1} (-1)^{i-k},
$$

which is just a rewriting of the desired identity (2).

Remark. It is unclear whether the second expression for u_d is achievable without the help of the variant (3) of Euler–Poincaré equation. If it was, then the whole argument could be separated from any form of Euler–Poincaré equation (see also a comment on [8, page 121]), as one would replace in (5) the identity $\sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} = (1+(-1)^{n-1}) f_{-1}$ by a definition of f_{-1} via this very identity. Defining f_{-1} in this way is indeed valid as soon as one notices that $\sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell} = (-1)^{n-1} \sum_{\ell=1}^{n} (-1)^{\ell-1} f_{n-\ell}$, which can be seen by evaluating at $d = 0$ the polynomial identity in the variable d generated from Lemma 1.

1.2 Connection with shelling parameters

Let us now imagine that the simplicial polytope S is constructed by placing one facet at a time. For $k \in [\![0:n]\!]$, we consider the number α_k of facets touching exactly k of the preceding facets. If $\alpha_0 = 1$, then the numbers $\alpha_1, \ldots, \alpha_n$ do not depend on the way S is constructed — they are called shelling parameters. Indeed, as a consequence of (9) below, taking $\alpha_0 = 1$ implies

$$\alpha_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{k-i} f_{i-1}, \qquad k \in [\![1:n]\!], \tag{7}$$

where the convention $f_{-1} = 1$ has again been used. Observe that, with $\alpha_0 = 1$, the case $k = n$ reads $\alpha_n = (-1)^n + \sum_{i=1}^{n} (-1)^{n-i} f_{i-1}$, so that the Euler-Poincaré equation (1) for a simplicial polytope is equivalent to $\alpha_n = 1$. The relations (7) follow from counting the domain points using yet another viewpoint, precisely

$$u_d = \sum_{k=0}^{n} \alpha_k \binom{d-k+n-1}{n-1}.$$

This gives a third expression for the generating function $G(z)$, namely

$$\begin{aligned} G(z) &= \sum_{d \geq 1} \sum_{k=0}^{n} \alpha_k \binom{d-k+n-1}{n-1} z^d = \alpha_0 \sum_{d \geq 1} \binom{d+n-1}{n-1} z^d + \sum_{k=1}^{n} \alpha_k \sum_{d \geq k} \binom{d+n-1-k}{n-1} z^k \\ &= \alpha_0 \left(\frac{1}{(1-z)^n} - 1 \right) + \sum_{k=1}^{n} \alpha_k z^k \sum_{j \geq 0} \binom{j+n-1}{n-1} z^j = \alpha_0 \left(\frac{1}{(1-z)^n} - 1 \right) + \sum_{k=1}^{n} \alpha_k \frac{z^k}{(1-z)^n} \\ &= \sum_{k=0}^{n} \alpha_k \frac{z^k}{(1-z)^n} - \alpha_0. \end{aligned}$$

Equating the latter with (4) and rearranging yields

$$\sum_{k=0}^{n} \alpha_k z^k = \alpha_0 (1-z)^n + \sum_{k=0}^{n-1} f_k z^{k+1} (1-z)^{n-k-1} = \alpha_0 (1-z)^n + \sum_{i=1}^{n} f_{i-1} z^i (1-z)^{n-i}. \tag{8}$$

Now, identifying the coefficient of z^k gives the generalization of (7) mentioned above, namely

$$\alpha_k = (-1)^k \alpha_0 \binom{n}{k} + \sum_{i=1}^{k} (-1)^{k-i} f_{i-1} \binom{n-i}{k-i}, \qquad k \in [\![1:n]\!]. \tag{9}$$

Remark. The sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ with $\alpha_0 = 1$ is also called the h-vector of S. The Dehn–Sommerville equations are often stated as $h_{n-k} = h_k$ for all $k \in [\![0:n]\!]$. The previous considerations make this fact apparent, too, since the Dehn–Sommerville equations are equivalent (see (6)) to

$$\sum_{k=-1}^{n-1} f_k \left(\frac{z}{1-z} \right)^{k+1} = (-1)^n \sum_{i=-1}^{n-1} (-1)^{i+1} f_i \left(\frac{1}{1-z} \right)^{i+1} = (-1)^n \sum_{i=-1}^{n-1} f_i \left(\frac{1/z}{1-1/z} \right)^{i+1}.$$

Taking (8) into account in the form $\sum_{j=-1}^{n-1} f_j(x/(1-x))^{j+1} = \left(\sum_{k=0}^{n} \alpha_k x^k\right)/(1-x)^n$, which is applied to $x = z$ and $x = 1/z$, the Dehn–Sommerville equations are seen to be equivalent to

$$\frac{\sum_{k=0}^{n} \alpha_k z^k}{(1-z)^n} = (-1)^n \frac{\sum_{k=0}^{n} \alpha_k (1/z)^k}{(1-1/z)^n} = \frac{\sum_{k=0}^{n} \alpha_k z^{n-k}}{(1-z)^n},$$

hence the equivalence with the formulation $\alpha_k = \alpha_{n-k}$ for all $k \in [\![0:n]\!]$.

2 Generating dimension formulas for spline spaces

In this section, Δ_n represents a simplicial partition of a domain $\Omega \subseteq \mathbb{R}^n$. The vector space of $\mathcal{C}^r$ splines of degree $\leq d$ over Δ_n is denoted by

$$\mathcal{S}_d^r(\Delta_n) := \left\{ s \in \mathcal{C}^r(\Omega) : s_{|T} \text{ is an } n\text{-variate polynomial of degree} \leq d \text{ for all } T \in \Delta_n \right\}.$$

We are interested in the dimension of this space. Finding a general formula is a notoriously difficult problem, as the partition Δ_n affects the dimension not only through its combinatorics but also through its geometry. Even the case of bivariate continuous splines is not settled, as formulas for $\dim \mathcal{S}_3^1(\Delta_2)$ and for $\dim \mathcal{S}_2^1(\Delta_2)$ remain to be established (see [3]). The case of trivariate splines seems even harder, because of its connection to the open Segre–Harbourne–Gimigliano–Hirschowitz conjecture (see [19]) and also because a solution to the trivariate problem would automatically settle the bivariate problem. Our viewpoint here is different: we specify a fixed simplicial partition Δ_n and we want to produce a formula for $\dim \mathcal{S}_d^r(\Delta_n)$ in a somewhat automated fashion. In [11], a method to do so was proposed based on a fundamental fact from Algebraic Geometry and on the use of Alfeld's applet [1] to compute $\dim \mathcal{S}_d^r(\Delta_n)$ for fixed d and r. Without reproducing the details, we only wish to point out that Bernstein–Bézier techniques, too, allow one to derive the fundamental fact which stipulates the existence of a integer $d^\star$ (depending on Δ_n and r and taken as small as possible) such that, if $d \geq d^\star$, then

$$\dim \mathcal{S}_d^r(\Delta_n), \text{ as function of } d, \text{ agrees with a polynomial in } d \text{ of degree } n.$$

This polynomial (depending on Δ_n and r) is called Hilbert polynomial. When $n = 2$, we can state this fact together with the bounds $d^\star \leq 4r+1$ for a general triangulation and, as a consequence of [13], with the bound $d^\star \leq 3r+2$ for a shellable triangulation. When $n = 3$, [5] yields the bound $d^\star \leq 8r+1$ for a general tetrahedration. For arbitrary n, the bound $d^\star \leq 2^n r + 1$ is anticipated but, to the best of our knowledge, unproved (although it holds for supersplines, see [6]). The fundamental fact can equivalently be stated in terms of the generating function of the sequence $(\dim \mathcal{S}_d^r(\Delta_n))_{d \geq 0}$ — the so-called Hilbert series — as the existence of a univariate polynomial P (depending on Δ_n and r) such that

$$\sum_{d \geq 0} \dim \mathcal{S}_d^r(\Delta_n)\, z^d = \frac{P(z)}{(1-z)^{n+1}}. \tag{10}$$

The equivalence is a direct consequence of the following observation applied to $u_d = \dim \mathcal{S}^r_d(\Delta_n)$ (a justification, found e.g. in [11], would also reveal that $\deg(P) = d^\star + n$):

Let $(u_d)_{d \ge 0}$ be a sequence for which there is a polynomial Q of degree m such that $u_d = Q(d)$ whenever $d \ge \bar{d}$ for some $\bar{d}$. Then there exists a polynomial R such that

$$\sum_{d \ge 0} u_d \, z^d = \frac{R(z)}{(1-z)^{m+1}}.$$

Furthermore, this observation provides a Bernstein–Bézier explanation of the additional information about P given in [7], namely that it has integer coefficients and that

$$P(1) = F_n, \qquad P'(1) = (r+1) F^{\text{int}}_{n-1}, \tag{11}$$

where F_n is the number of simplices of Δ_n and F^{int}_{n-1} is the number of interior facets of Δ_n. Indeed, the lower and upper bounds derived in [2] imply that

$$\dim \mathcal{S}^r_d(\Delta_n) = F_n \binom{d+n}{n} - (r+1) F^{\text{int}}_{n-1} \binom{d+n-1}{n-1} + \mathcal{O}(d^{n-2}).$$

Then (11) is obtained by applying the above observation to the quantity

$$u_d = \dim \mathcal{S}^r_d(\Delta_n) - F_n \binom{d+n}{n} + (r+1) F^{\text{int}}_{n-1} \binom{d+n-1}{n-1},$$

which reduces to a polynomial of degree $\le n-2$ when d is large enough. The fact that P has integer coefficients can be seen by looking at the coefficient a_k of z^k in $P(z) = (1-z)^{n+1} \sum_{d \ge 0} \dim \mathcal{S}^r_d(\Delta_n) \, z^d$, which incidentally shows that $a_0, a_1, a_2, \ldots$ can be deduced one at a time by computing $\dim \mathcal{S}^r_0(\Delta_n)$, $\dim \mathcal{S}^r_1(\Delta_n)$,$\dim \mathcal{S}^r_2(\Delta_n), \ldots$ sequentially. Once P has been completely determined via the values of $a_0, a_1, \ldots, a_{d^\star+n}$, the identity (10) leads to the dimension formula

$$\dim \mathcal{S}^r_d(\Delta_n) = \sum_{k=0}^{d^\star+n} a_k \binom{d+n-k}{n}.$$

This way (one among many!) of expressing the dimension formula hides the dependence on the geometry in the coefficients a_k's. In [11], we exploited the method just outlined to conjecture formulas for several specific partitions. The one for the Alfeld split with arbitrary n has been fully justified since then (see [18]), and we now wish to stimulate investigations concerning octahedra (regular and generic) in the case $n = 3$. We point out that our formulas remained at the conjecture level because of some guesswork involved in the process: uncertainty about the required number of dimension computations (the anticipated bound on $d^\star$ is often too large) and unknown behavior of the a_k's as a function of r (and of n when appropriate). Although the first issue can be lifted via a direct computation of the Hilbert series — using e.g. the conversion to Commutative Algebra proposed in [9] by P. Clarke and implemented in SAGE as a package called SplineDim (downloadable on the author's webpage) — nothing, to the best of our knowledge, can yet be said about the second issue.

3 Reformulations of Schumaker's partial interpolation conjecture

Given a triangle T with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and given an integer $d \geq 1$, let

$$\mathcal{D}_d := \left\{ \xi_{i,j,k} := \frac{i}{d}\mathbf{v}_0 + \frac{j}{d}\mathbf{v}_1 + \frac{k}{d}\mathbf{v}_2 : i \geq 0, j \geq 0, k \geq 0, i+j+k=d \right\}$$

be the set of $\binom{d+2}{2}$ domain points of order d relative to T. Associated to each domain point, there is a bivariate Bernstein polynomial defined for $\mathbf{v} \in T$ by

$$B_{\xi_{i,j,k}}(\mathbf{v}) = \frac{d!}{i!j!k!} x^i y^j z^k,$$

where (x, y, z) are the barycentric coordinates of $\mathbf{v}$, so that $\mathbf{v} = x\,\mathbf{v}_0 + y\,\mathbf{v}_1 + z\,\mathbf{v}_2$ with $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + y + z = 1$. In 2003, Schumaker raised the conjectured stated in [4] that, for any subset Γ of $\mathcal{D}_d$, every dataset of $\gamma = \mathrm{card}(\Gamma)$ values can be uniquely interpolated at Γ by a function from $\mathrm{span}\{B_\xi, \xi \in \Gamma\}$. In other words, the linear map: $s \in \mathrm{span}\{B_\xi, \xi \in \Gamma\} \mapsto (s(\eta), \eta \in \Gamma) \in \mathbb{R}^\gamma$ was conjectured to be invertible. In matrix form, the question to be answered (which is in fact independent of the triangle T) reads:

(12) $$\text{Is } \mathbf{B}_{\Gamma,\Gamma} := [B_\xi(\eta)]_{\eta,\xi\in\Gamma} \text{ an invertible matrix for any } \Gamma \subseteq \mathcal{D}?$$

This is a weak formulation of the conjecture, and the strong formulation reads:

(13) $$\text{Is } \det(\mathbf{B}_{\Gamma,\Gamma}) > 0 \text{ for any } \Gamma \subseteq \mathcal{D}?$$

Using a different vocabulary, this strong formulation asks exactly if the whole matrix $\mathbf{B}_{\mathcal{D}_d,\mathcal{D}_d}$ is a P-matrix. Requiring the determinant to be positive seems natural because this is the case for $\gamma = 1$ (i.e., $B_\xi(\xi) > 0$ for all $\xi \in \mathcal{D}$), for $\gamma = 2$ (i.e., $B_\xi(\xi)B_\eta(\eta) - B_\eta(\xi)B_\xi(\eta) > 0$ since any B_ζ takes it maximum value at ζ), and for $\gamma = \binom{d+2}{2}$ (i.e., the product of the eigenvalues of $\mathbf{B}_{\mathcal{D}_d,\mathcal{D}_d}$ is positive since [10] showed that each eigenvalue is positive). One can observe that the strong formulation of the conjecture is true up to $d \leq 17$ as follows (this observation was independently made and published in [14]): first, notice that one only needs $\mathbf{B}_{\mathcal{D}'_d,\mathcal{D}'_d}$ to be a P-matrix, where

$$\mathcal{D}'_d := \left\{ \xi_{i,j,k} := \frac{i}{d}\mathbf{v}_0 + \frac{j}{d}\mathbf{v}_1 + \frac{k}{d}\mathbf{v}_2 : i \geq 1, j \geq 1, k \geq 1, i+j+k=d \right\}$$

is the set of interior domain points of order d, because

$$\mathbf{B}_{\mathcal{D}_d,\mathcal{D}_d} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{X} & \mathbf{B}_{\mathcal{D}'_d,\mathcal{D}'_d} \end{array}\right] \quad \text{for some totally positive matrix } \mathbf{A};$$

then, remark that $\mathbf{B}_{\mathcal{D}'_d,\mathcal{D}'_d}$ is a P-matrix as soon as the symmetric matrix $\mathbf{C} := c_d \left(\mathbf{B}_{\mathcal{D}'_d,\mathcal{D}'_d} + \mathbf{B}^\top_{\mathcal{D}'_d,\mathcal{D}'_d} \right)$ is positive definite (we choose $c_d = \exp(d)$); finally, prove numerically that $\mathbf{C}$ is positive definite for all $d \leq 17$ by establishing that its minimum eigenvalue is positive. A similar strategy can be called upon to derive that the trivariate version of the strong conjecture holds up to $d = 16$ and that the quadrivariate version holds up to $d = 14$ (see the MATLAB reproducible file available on the author's webpage).

It is now time to showcase three reformulations of Schumaker's conjecture in order to offer some other angles of attack. The first one simply translates a known characterization of P-matrices in terms of the linear complementarity problem (see [17] and also [16] for corrections to further characterizations), so that the strong version (13) can be phrased as:

Does there exists, for every $\mathbf{q} \in \mathbb{R}^n$, an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \geq \mathbf{0}$, $\mathbf{M}\mathbf{x}+\mathbf{q} \geq \mathbf{0}$, and $\langle \mathbf{M}\mathbf{x}+\mathbf{q}, \mathbf{x} \rangle = 0$? Here $\mathbf{M} = \mathbf{B}_{\mathcal{D}_d,\mathcal{D}_d}$ and $n = \binom{d+2}{2}$ or $\mathbf{M} = \mathbf{B}_{\mathcal{D}'_d,\mathcal{D}'_d}$ and $n = \binom{d-1}{2}$.

The second reformulation is phrased in terms of a decomposition of the space $\mathcal{P}_d$ of polynomials of degree $\leq d$ in two variables u and v. Precisely, the weak version (12) can be phrased as:

Does $\big\{[d - i(1-u) - j(1-v)]^d, (i,j) \in \Lambda\big\} \cup \big\{u^i v^j, (i,j) \in \mathcal{I}_d \setminus \Lambda\big\}$ form a basis for the space $\mathcal{P}_d$ whatever the subset Λ of $\mathcal{I}_d := \{(i,j) : i \geq 0, j \geq 0, i+j \leq d\}$?

This reformulation results from the observation that, for any $(i,j) \in \mathcal{I}_d$ and any $\mathbf{v} \in T$ with barycentric coordinates $(x, y, 1-x-y)$,

$$B_{\xi_{i,j,d-i-j}}(\mathbf{v}) = \frac{1}{i!j!} \frac{\partial^{i+j}[1 - x(1-u) - y(1-v)]^d}{\partial u^i \partial v^j}(0,0).$$

The weak version stating that, for any $\Lambda \subseteq \mathcal{I}$, the equalities $\sum_{(\mu,\nu)\in\Lambda} c_{\mu,\nu} B_{\xi_{i,j,d-i-j}}(\xi_{\mu,\nu,d-\mu-\nu}) = 0$ for all $(i,j) \in \Lambda$ imply that $c_{\mu,\nu} = 0$ for all $(\mu,\nu) \in \Lambda$, can therefore be rephrased as the following statement: for any $\Lambda \subseteq \mathcal{I}$ and any $p(u,v) := \sum_{(\mu,\nu)\in\Lambda} c_{\mu,\nu} \left[1 - (\mu/d)(1-u) - (\nu/d)(1-v)\right]^d$,

$$\Big(\frac{1}{i!j!} \frac{\partial^{i+j} p}{\partial u^i \partial v^j}(0,0) = 0 \text{ for all } (i,j) \in \Lambda \Big) \Rightarrow \Big(c_{\mu,\nu} = 0 \text{ for all } (\mu,\nu) \in \Lambda \Big).$$

Since the basis $\big(p \in \mathcal{P}_d \mapsto \frac{1}{i!j!}\frac{\partial^{i+j} p}{\partial u^i \partial v^j}(0,0) \in \mathbb{R}, (i,j) \in \mathcal{I}_d\big)$ of linear functionals on $\mathcal{P}_d$ is dual to the basis $\big(u^i v^j, (i,j) \in \mathcal{I}_d\big)$ of $\mathcal{P}_d$, the weak version can be rephrased as: for any $\Lambda \subseteq \mathcal{I}$,

$$\Big(p \in \operatorname{span}\big\{ \left[1 - (\mu/d)(1-u) - (\nu/d)(1-v)\right]^d, (\mu,\nu) \in \Lambda \big\} \cap \operatorname{span}\big\{ u^i v^j, (i,j) \in \mathcal{I}_d \backslash \Lambda \big\} \Big) \Rightarrow \Big(p = 0 \Big).$$

In other words, for any $\Lambda \subseteq \mathcal{I}$, the above two subspaces of $\mathcal{P}_d$ are in direct sum, and because their dimensions add up to the dimension of $\mathcal{P}_d$, their direct sum is in fact equal to the whole space $\mathcal{P}_d$. The statement is now easily translated into the second reformulation of the conjecture.

The third reformulation involves a bivariate Vandermonde matrix at points $(x_{i,j}, y_{i,j}) \in \mathbb{R}^2$ indexed by $(i,j) \in \mathcal{I}$ and occurring as intersections of three families of $d+1$ lines each (in which case the fundamental Lagrange interpolators can be expressed explicitly on the model of [15, p.12]). Precisely, the strong version (13) can be phrased as:

Is $\det \left[x^{\mu}_{i,j} y^{\nu}_{i,j}\right]_{(i,j),(\mu,\nu)\in\Lambda} > 0$ *for every subset* Λ *of* $\mathcal{I}_d$*, with* $x_{i,j} := \dfrac{i}{d-i-j}$ *and* $y_{i,j} := \dfrac{j}{d-i-j}$*?*

After identifying domain points $\xi_{\mu,\nu,d-\mu-\nu} \in \Gamma$ with indices $(\mu,\nu) \in \Lambda$, the above reformulation results from manipulating a determinant by factoring along rows and columns as follows (the shorthand $\backsim$ means 'has the sign of'):

$$\det\left[B_\xi(\eta)\right]_{\eta.\xi\in\Gamma} = \det\left[\frac{d!}{\mu!\nu!\kappa!}\left(\frac{i}{d}\right)^{\mu}\left(\frac{j}{d}\right)^{\nu}\left(\frac{k}{d}\right)^{\kappa}\right]_{(\mu,\nu),(i,j)\in\Lambda} \backsim \det\left[i^{\mu} j^{\nu} (d-i-j)^{d-\mu-\nu}\right]_{(\mu,\nu),(i,j)\in\Lambda}$$
$$\backsim \det\left[\left(\frac{i}{d-i-j}\right)^{\mu}\left(\frac{j}{d-i-j}\right)^{\nu}\right]_{(\mu,\nu),(i,j)\in\Lambda} = \det\left[x^{\mu}_{i,j} y^{\nu}_{i,j}\right]_{(\mu,\nu),(i,j)\in\Lambda}.$$

We finally remark that a point $(x_{i,j}, y_{i,j})$ with $(i,j) \in \mathcal{I}$ belongs to the lines L_i, M_j, and N_k, $k := d-i-j$, where $L_0,\ldots,L_d$, $M_0,\ldots,M_d$, and $N_0,\ldots,N_d$ have equations

$$L_i : (d-i)x - iy = i, \qquad M_j : -jx + (d-j)y = j, \qquad N_k : kx + ky = d-k.$$

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